Converse Propositions for the Mean and Average Value Theorems of Calculus

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Sometime around 2018, as a Calculus teacher I was looking up Cauchy's integral formula in a complex analysis textbook, curiously I tried to make up a "real" version of the theorem suitable for students of Calculus. It turned out that the real version of the formula can be interpreted as a converse to the Mean Value Theorem for integrals in Calculus. And since I had been teaching Calculus at different levels for decades by then, I was also surprised this educational fact hadn't been mentioned in any of the many Calculus textbooks I had dealt with. So, within a reasonable time was able to use the available tools of Calculus to prove the converse, as seen in the Proposition below. Moreover, once the proof of the converse was settled, it also turned out that one can use the Proposition to conclude the converse for the Mean Value Theorem for differentiable functions, as seen in corollary following the Proposition. I should also point out that a certain condition (that is $f'(c) \neq 0$) must hold for the proof of the Proposition, therefore the Remark following the Proposition brings a counter example, showing that the said condition must necessarily be me for the assertion of the Proposition to be valid. I should also mention that I have posted my finding in this short article for BC math teachers on the bcamt's group email perhaps around late 2018. Finally, Lemm 1following the Corollar shows the assertion of the Corollary will not hold (at least locally) if the point $(c, f(c))$ is an inflection point for $f(x)$.

Proposition (The converse of the Mean Value Theorem for integrals)
Let $y = f(x)$ be a continuously differentiable function over an open interval I. Then for any number c in I for which $f'(c) \neq 0$ the value $f(c)$ of the function can be expressed as an average value for $f(x)$ in terms of a definite integral in the form $f(c) = \frac{1}{b-a}\int_a^b f(t)dt$, with $a, b \in I$.

Proof As I will explain shortly, we can assume without loss of generality $c = 0$ and $f(c) = 0$ at the same time. In this more convenient case, the assumption $f'(0) \neq 0$ implies either $f'(0) > 0$ or else $f'(0) < 0$. Because of similarity of the argument for the two cases I will assume $f'(0) > 0$. Then, since the derivative $f'(x)$ of f is assumed to be continuous at $c = 0$, there will exist an open subinterval (a, b_1) of I containing $c = 0$ such that $f'(x) > 0$ throughout the interval (a, b_1), which in turn implies $f(x)$ is increasing over (a, b_1). Therefore, for the specific anti-derivative of $f(x)$ defined by $F(x) = \int_0^x f(t)dt$, $x \in [a, b_1]$ we will have

$$F(a) = \int_0^a f(t)dt = -\int_a^0 f(t)dt > 0 \text{ , and } F(b_1) = \int_0^{b_1} f(t)dt > 0.$$

Now if the two positive numbers $F(a)$ and $F(b_1)$ are equal we are done, because

$$\frac{1}{b_1 - a}\int_a^{b_1} f(t)dt = \frac{1}{b-a}[\int_a^0 f(t)dt + \int_0^{b_1} f(t)dt]\frac{1}{b_1 - a}[F(b_1) - F(a)] = 0 = f(0).$$

Otherwise, one of the two positive numbers $F(a)$ and $F(b_1)$ is greater than the other one, say $0 < F(a) < F(b_1)$. Then, since the anti-derivative function $F(x)$ is continuous over the interval $[0, b_1]$ and we also have the inequality $F(0) = 0 < F(a) < F(b_1)$, by the intermediate value theorem for continuous functions applied to $F(x)$ over the interval $[0,b_1]$ there will exist a positive number $0<b<b_1$ such that $F(b)=F(a)=-\int_a^0 f(t)dt$. Therefore, we will have

$$\frac{1}{b-a}\int_a^b f(t)dt=\frac{1}{b-a}\int_a^b f(t)dt=\frac{1}{b-a}[\int_a^0 f(t)dt+\int_0^b f(t)dt]$$
$$=\frac{1}{b-a}[-F(a)+F(b)]=0=f(0)\,,$$

and the proof for the convenient case of $c=0$ and $f(c)=0$ is complete.

If to begin with $c\neq 0$ and/or $f(c)\neq 0$, we will first apply the above argument to the function $g(x)=f(x+c)-f(c)$ whose domain is the interval the interval $J = I - c$ with $,0\in J,$ and $g(0)=0$. Therefore, by the previous case we have proved there will be numbers $u<0<v$, $u,v\in J,$ such that the integral relation $g(0)=\frac{1}{v-u}\int_u^v g(t)dt$ holds. Next, considering the relation $g(x)=f(x+c)-f(c)$ between f and g, the latter integral relation in terms of f translates as

$$0=g(0)=\frac{1}{v-u}\int_u^v [\,f(t+c)-f(c)\,]dt$$

If we now separate the integral on the right into two parts and bring the second part to the left we get

$$f(c)=\frac{f(c)}{v-u}\int_u^v dt=\frac{1}{v-u}\int_u^v f(t+c)\,dt$$

If we make a substitution $x=t+c$ in the integral on the right we get

$$f(c)=\frac{1}{v-u}\int_{u+c}^{v+c} f(x)dx$$

Next, since the $u,v\in J$ and $u<0<v$ imply $u+c, v+c\in I$, and $u+c<c<v+c$ respectively, setting a = $u+c$ and $b = v+c$ tunes the above last integral into

$$f(c)=\frac{1}{(b-c)-(a-c)}\int_a^b f(x)dx=\frac{1}{(b-a)}\int_a^b f(x)dx$$

and the proof of the general case is also complete.

Remarks:(a) The condition $f'(c)\neq 0$ in the Proposition is essential. For example, the assertion of the Proposition isn't valid for $f(x)=x^2+1$ and $c=0$.

(b) The assumption of *continuous differentiability* of $f(x)$ in the above Proposition may be somewhat weakened. One can conclude the same "converse" assertion by merely assuming that

$f(x)$ is a *strictly monotonic continuous* function over the interval I, in which case only the first few first lines of the above proof need to be shortened a little, but the remaining majority part of the proof will stay unchanged.

(c) The integral relation $f(c) = \frac{1}{b-a}\int_a^b f(t)dt$ of the Proposition can be expressed in a form consistent with the polar form of the Cauchy's Integral Formula in complex analysis as,

$$f(c) = \frac{1}{b-a}\int_a^b f(t)dt = \frac{1}{2\pi}\int_0^{2\pi} f(a+\lambda\theta)d\theta$$

with $\lambda = (b-a)/2\pi$. To show the extent of consistency of the above, I recall that the polar version of Cauchy's Integral Formula is as follows,

$$f(c) = \frac{1}{2\pi}\int_0^{2\pi} f(c+re^{i\theta})d\theta,\ 0 < r < d(c;\partial D).$$

Corollary: (The converse of the mean value theorem)

Let $y = g(x)$ be a twice differentiable function over an open interval I containing a number c, and let $g''(c) \neq 0$. Then there is a secant segments with end-points $A(a, g(a))$, $B(b, g(b))$ parallel to the tangent line T_c at the point $P(c, g(c))$ supported by the graph of $y = g(x)$, with $a, b \in I$, $a < b$ which is to the curve $y = g(x)$. More precisely, there will exist numbers $a, b \in I$ with $a < b$ such that

$$g'(c) = \frac{g(b)-g(a)}{b-a}\ .$$

Proof: Simply choose $f(x) = g'(x)$ and apply the above Proposition. Thus, by the Fundamental Theorem of Calculus, there will exist $a, b \in I$, $a < b$ such that

$$g'(c) = f(c) = \frac{1}{b-a}\int_a^b f(t)dt = \frac{1}{b-a}\int_a^b g'(t)dt = \frac{g(b)-g(a)}{b-a}$$

We end this short article with the following two Lemma relevant to the above corollary.

Lemma 1 Let $f(x)$ be a twice differentiable function over the real line, and let $P(c, \text{f}(c))$ be an inflection at which $f(x)$ changes concavity. That is, $f''(\text{c}) = 0$ and also $f(x)$ changes concavity at P. Then there is an open interval $I = (a, b)$ containing c such that for any $u, v \in I$ satisfying $u < c < v$ the inequality

$$f'(c) \neq \frac{f(v)-f(u)}{v-u}$$

holds. Otherwise said, if $u, v \in I$ and $u < c < v$, no secant segment UV, with end points $U(u, f(u)$ and $V(v, f(v)$ will be parallel to the tangent line T_c at the inflection point $P(c, \text{f}(c))$ to the graph of $f(x)$.

Proof: Since the assertion of the Lemma is invariant under any horizontal or vertical transformation of $f(x)$, we can assume without loss of generality that the inflection point is at the origin, that is $P(c, \text{f}(c))$ is the origin O (0,0), in which case we will have $f(c) = c = 0$.
Since $f(x)$ changes concavity at the origin, there is a positive δ such that $f(x)$ changes concavity only once at $c = 0$ inside the interval I = (- δ δ). To complete the proof we will assume otherwise and reach a contradiction. So, assume there are two real numbers $u\ and\ v$ satisfying $-\delta < u < 0 < v < \delta$ such that

$$f'(0) = \frac{f(v)-f(u)}{v-u}. \qquad (*)$$

Now because of similarity of the arguments we will assume that $f(x)$ is concave down under the interval $[-\delta, 0)$ and concave up over the interval $(0,\delta]$. Since the graph of $f(x)$ is below the tangent line T_0: $y = f'(0)$ x below the interval, $[-\delta, 0)$ and $u \in [-\delta, 0)$, it follows that $f(u) < f'(0)\, u$, which could also be expressed as $-f'(0)\, u < -f(u)$. On the other hand, since the graph of $f(x)$ is above the tangent line T_0: $y = f'(0)$ x and $v \in (0, \delta]$, over the interval $(0, v]$, we should also have $f'(0)\, v < f(v)$.

Now, adding the corresponding sides of the latter it two inequalities we get $f'(0)\, v - f'(0)\, u < f(v) - f(u)$. This inequality in turn implies

$$f'(0 = \frac{f(v)-f(u)}{v-u},$$

contradicting the assumption (*), and the proof of the Lemma is completed.

We end the article with the following obvious Lemma

Lemma 2 Let $f(x)$ be a twice differentiable function over the real line, and let $P(c, \mathrm{f}(c))$ be an inflection point at which $f(x)$ changes concavity, so that hat is $f''(c) = 0$. Then the necessary and sufficient condition for any two real numbers u and v with $u < c < v$ to satisfy the relation

$$f'(c) = \frac{f(v)-f(u)}{v-u}$$

is that either for some $u < c$ the equation $f(x) = f'(c)\,(x - u) + f(u)$ has a solution $x = v$ with $c < v$; or for some $c < v$ the equation $f(x) = f'(c)\,(x - v) + f(v)$ has a solution $x = u$ with $u < c$.

Proof Since both the lines with the equations $y = f'(c)\,(x - u) + f(u)$ and $y = f'(c)\,(x - v) + f(v)$ are parallel to the tangent line T_c, at $P(c, \mathrm{f}(c))$ and in both cases $u < c < v$, the proof is straightforward.