

# A Method of Implicit Integration

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## Abstract

*In this article we introduce a method that enable us to find antiderivatives for differentiable functions defined through a specific class of implicit relations  $R(x, y) = 0$  on the  $xy$ - coordinate system. The method works for the class of relations for which upon polar substitutions  $x = \rho \cos \theta$  and  $y = \rho \sin \theta$  the resulting polar relation  $P(\rho, \theta) = 0$  can be solved for the variable  $\rho$  in terms of  $\theta$  and that the trigonometric function  $\rho^2(\theta)$  is integrable in terms of  $\theta$ .*

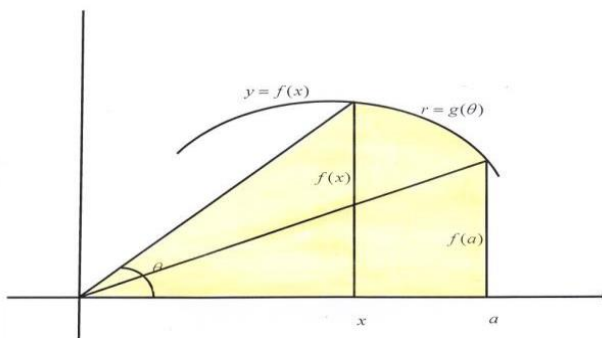
As far as the author's previous decades of teaching Calculus at all levels is concerned, the method might be the closest one can get to integrate an implicitly defined function, yet. The author came up with the method early 2016 while teaching AP-Calculus at Prince of Wales Secondary in Vancouver, and a posted the idea through examples on BCAMT email list serve on April 2016 to share it with AP-Calculus teachers. The method has not been mentioned in any classic Calculus textbooks the author has seen, so it might sound reasonable to assume that the method is has not been explored before. Here, we first start by stating the main Theorem introducing the method, and then bring 3 Example to show how the method works for concrete relations  $R(x, y) = 0$ .

**Theorem** [A. Astaneh] Let  $y = f(x)$  be a real valued continuous function implicitly defined by a relation  $R(x, y) = 0$ , and let  $P(\rho, \theta) = 0$  be the polar version of the relation obtained upon substitution of  $x = \rho \cos \theta$  and  $y = \rho \sin \theta$  in the relation  $R = 0$ . Furthermore, assume that the latter polar relation can be solved for  $\rho$  in terms of  $\theta$ , and that the trigonometric function  $\rho(\theta)^2$  is integrable in terms of  $\theta$ . Then for any real number  $a$  in the domain of  $f(x)$ , the function defined by

$$F(x) = \frac{x f(x)}{2} - \frac{1}{2} \int_{\tan^{-1}(\frac{f(a)}{a})}^{\tan^{-1}(\frac{f(x)}{x})} \rho(\theta)^2 d\theta - \frac{af(a)}{2} \quad (1)$$

Is an antiderivative for  $f(x)$  with initial condition  $F(a) = 0$ .

Note that here  $y = f(x)$ , and considering the relations  $\sin \theta = \frac{y}{\sqrt{x^2+y^2}}$ ,  $\cos \theta = \frac{x}{\sqrt{x^2+y^2}}$ , and  $\theta = \tan^{-1}(\frac{y}{x})$ , the definite integral  $\int_{\tan^{-1}(\frac{f(a)}{a})}^{\tan^{-1}(\frac{f(x)}{x})} \rho(\theta)^2 d\theta$ , and hence the antiderivative  $F(x)$  will eventually be expressed in terms of general  $x$  and  $y$ .



Note that, considering the relations  $\sin \theta = \frac{y}{\sqrt{x^2+y^2}}$ ,  $\cos \theta = \frac{x}{\sqrt{x^2+y^2}}$ , and  $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ , the function  $F(x)$  will eventually be expressed in terms of general  $x$  and  $y$ .

We first show how formula (1) comes about as an antiderivative for  $f(x)$ , and then use the Leibniz Integral Rule (a combination of the Fundamental Theorem of Calculus and the chain Rule) to present an Analytical proof. bring a proof to reaffirm that  $F(x)$  in (1) is indeed an antiderivative for  $y = f(x)$ .

**Geometrical Proof:** We first show how the formula (1) comes about to be an antiderivative for  $f(x)$ , and then use the Leibniz Integral Rule to bring a proof independent from above figure to reaffirm that  $F(x)$  define by (1) is an antiderivative for  $y = f(x)$ . To this end, let  $a$  be a real number in the domain of  $f(x)$  defined by the elation  $R(x, y) = 0$ . Then for any given  $x$  other than  $a$  in the domain of  $f(x)$  consider the coloured area enclosed by the  $x$  –axis, the vertical line  $x = a$ , the graph of  $f(x)$ , and by the ray of line emanating from the origin and making an angle of  $\theta = \tan^{-1}\left(\frac{f(x)}{x}\right)$  with the positive  $x$  –axis. Now let  $F(x)$  be the antiderivative of  $f(x)$ . Then by Fundamental of Calculus the area directly above the interval  $[x, a]$  will be  $F(a) - F(x)$ . Also let  $A_\theta$  denote the polar area between the polar curve  $r = \rho(\theta)$  lying between the rays of lines defined by angles  $\theta_1 = \tan^{-1}\left(\frac{f(a)}{a}\right)$  and  $\theta = \tan^{-1}\left(\frac{f(x)}{x}\right)$ . Then, since the area of right triangle with the height  $f(x)$  seen above  $\frac{x f(x)}{2}$ , the described enclosed area can be represented by the two equal expressions  $\frac{x f(x)}{2} + F(a) - F(x) = \frac{a f(a)}{2} + A_p$ . Now using the popular formula  $\frac{1}{2} \int_{\theta_1}^{\theta} \rho(\theta)^2 d\theta$  for the polar area  $A_p$ , the above relation can be expressed as

$$\frac{x f(x)}{2} + F(a) - F(x) = \frac{a f(a)}{2} + \frac{1}{2} \int_{\tan^{-1}\left(\frac{f(a)}{a}\right)}^{\tan^{-1}\left(\frac{f(x)}{x}\right)} \rho(\theta)^2 d\theta.$$

Therefore, summing up all constant numbers on the right as  $C$ , it follows that

$$F(x) = \frac{x f(x)}{2} - \frac{1}{2} \int_{\tan^{-1}\left(\frac{f(a)}{a}\right)}^{\tan^{-1}\left(\frac{f(x)}{x}\right)} \rho(\theta)^2 d\theta - \frac{a f(a)}{2},$$

Is an antiderivative for  $f(x)$ , and therefore.

Since the above argument relies heavily on above diagram, one may ask for an analytical proof of (1) independent of the diagram, and here it is,

**Analytical Proof:** To this end, we use Leibniz Integral Rule (the fundamental theorem of calculus combined with the chain rule) to reassure that  $F'(x) = f(x) = y$ , and here it goes,

$$F'(x) = \frac{d}{dx} \left[ \frac{xy}{2} - \frac{1}{2} \int_{\tan^{-1}\left(\frac{f(a)}{a}\right)}^{\tan^{-1}\left(\frac{y}{x}\right)} \rho(\theta)^2 d\theta - \frac{a f(a)}{2} \right] =$$

$$\frac{y + xy'}{2} - \frac{1}{2} \frac{d}{dx} \left[ \int_0^{\tan^{-1}\left(\frac{y}{x}\right)} \rho(\theta)^2 d\theta - \int_0^{\tan^{-1}\left(\frac{f(a)}{x}\right)} \rho(\theta)^2 d\theta \rho \left( \tan^{-1}\left(\frac{y}{x}\right) \right) \right] =$$

$$\begin{aligned}
& \frac{y + xy'}{2} - \frac{1}{2} \left[ \rho \left( \tan^{-1} \left( \frac{y}{x} \right) \right)^2 \frac{d}{dx} \left( \tan^{-1} \left( \frac{y}{x} \right) \right) - \rho \left( \tan^{-1} \left( \frac{f(a)}{a} \right) \right)^2 \frac{d}{dx} \left( \tan^{-1} \left( \frac{f(a)}{a} \right) \right) \right] \\
& = \\
& = \frac{y + xy'}{2} - \frac{1}{2} \left[ \rho (\theta)^2 \frac{d}{dx} \left( \tan^{-1} \left( \frac{y}{x} \right) \right) \right] = \\
& \frac{y + xy'}{2} - \frac{1}{2} \left[ (x^2 + y^2) \times \frac{xy' - y}{x^2 + y^2} \right] = \frac{y + xy'}{2} - \frac{1}{2} (xy' - y) = y = f(x).
\end{aligned}$$

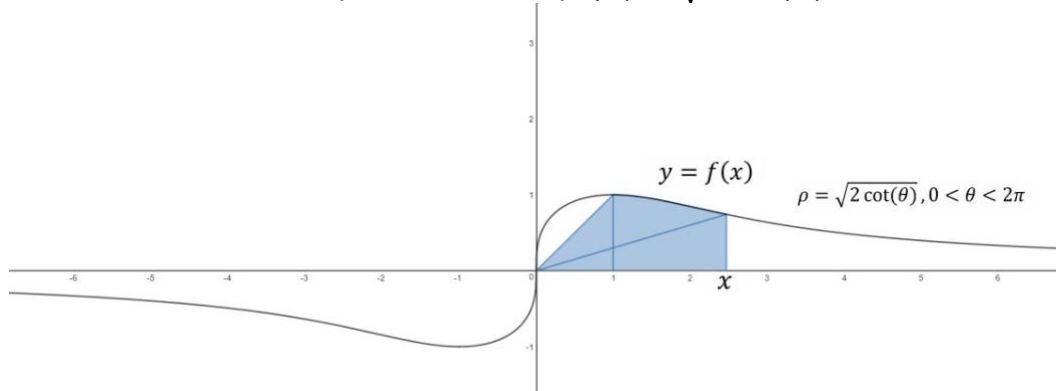
This completes the proof of the Theorem.

**Example 1** Let  $y = f(x)$  be the function implicitly defined by the relation

$$y(x^2 + y^2) = 2x, \quad x > 0, y > 0. \quad (2)$$

Find an anti-derivative  $F(x)$  for  $f(x)$  with initial condition  $F(1) = 0$ .

**Analytic Solution:** Upon polar substitutions  $x = \rho \cos \theta$  and  $y = \rho \sin \theta$  in relation (2), the relation is converted into  $\rho^2 = 2 \cos \theta$ , or  $\rho(\theta) = \sqrt{2 \cos(\theta)}$  with the following polar graph



Now let  $F(x)$  be the antiderivative of  $f(x)$ , defining the area under the graph directly above the interval  $[1, x]$ , and let  $A_x$  denote the same area. Also let  $A_\theta$  denote the polar area enclosed by the curve  $r = \rho(\theta)$  and the rays of lines defined by angles  $\frac{\pi}{4}$  (note that the point  $(1,1)$  is on the curve) and  $\theta = \tan^{-1} \left[ \frac{f(x)}{x} \right]$ . Then the above figure shows the validity of the relation between the two areas  $A_x$  and  $A_\theta$ ,

$$\frac{1}{2} f(1) + A_x = \frac{x f(x)}{2} + A_\theta.$$

Now, by Fundamental Theorem of Calculus, and the polar formula for the area  $A_\theta$ , the above relation implies

$$\frac{1}{2} + F(x) - F(1) = \frac{x f(x)}{2} + \frac{1}{2} \int_{\tan^{-1} \left[ \frac{f(a)}{a} \right]}^{\frac{\pi}{4}} \rho(\theta)^2 d\theta,$$

$$F(x) = \frac{x f(x)}{2} - \frac{1}{2} + \frac{1}{2} \int_{\tan^{-1}\left\{\frac{f(a)}{a}\right\}}^{\frac{\pi}{4}} 2 \cot(\theta) d\theta,$$

$$F(x) = \frac{x f(x)}{2} - \frac{1}{2} + \ln(\sin(\theta)) \Bigg|_{\tan^{-1}\left[\frac{f(a)}{a}\right]}^{\frac{\pi}{4}}$$

$$F(x) = \frac{x f(x)}{2} - \frac{1}{2} - \frac{1}{2} \ln(2) - \ln \frac{y}{\sqrt{x^2+y^2}}.$$

And therefore,

$$F(x) = \frac{x f(x)}{2} + \ln \frac{\sqrt{x^2+f(x)^2}}{\sqrt{2} f(x)} - \frac{1}{2}$$

is the antiderivative of the function  $y = f(x)$  implicitly defined by relation (1), with initial condition  $F(1) = 0$ .

**Direct Verification of the Solution:** It is enough to show that the derivative of  $F(x)$  as expressed above is  $y = f(x)$ . To this end, in the following manipulation we will make use of the original implicit relation (1), in the sense that when a denominator  $y(x^2 + y^2)$  show up in the process we will replace it with  $2x$ .

$$\begin{aligned} F'(x) &= \frac{d}{dx} \left[ \frac{xy}{2} + \ln \frac{\sqrt{x^2+y^2}}{\sqrt{2} y} - \frac{1}{2} \right] = \\ &= \frac{d}{dx} \left[ \frac{xy}{2} + \frac{1}{2} \ln(x^2 + y^2) - \ln 2 \right] = \\ &= \frac{y + xy'}{2} + \frac{2x + 2yy'}{2(x^2 + y^2)} - \frac{y'}{y} = \frac{y + xy'}{2} + \frac{y(x + yy') - y'(x^2 + y^2)}{y(x^2 + y^2)} \\ &= \frac{y + xy'}{2} + \frac{y(x + yy') - y'(x^2 + y^2)}{2x} = \\ &= \frac{xy + x^2 y' + xy + y^2 y' - x^2 y' - y^2 y'}{2x} = y = f(x). \end{aligned}$$

Note that the above verification is also a proof that

$$F(x) = \frac{x f(x)}{2} + \ln \frac{\sqrt{x^2+f(x)^2}}{\sqrt{2} f(x)} + C$$

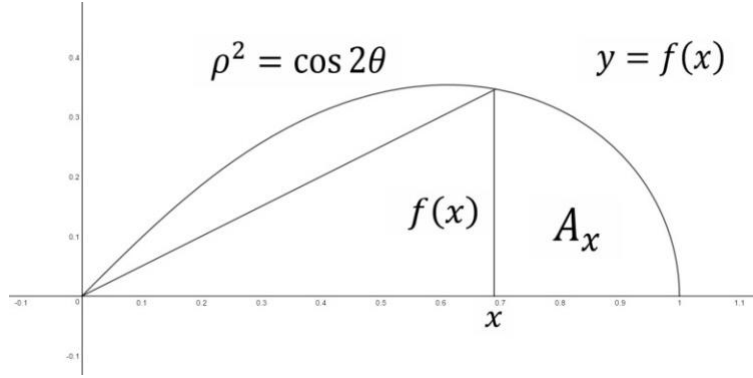
Is the general antiderivative for the implicitly defined function whose graph is the left part of the graph of the original cartesian relation, that is for  $g(x) = -f(-x)$ ,  $x < 0$ . It is left to the reader to find a particular antiderivative for  $g(x)$ .

**Example 2** Let  $y = f(x)$  be the function implicitly defined by the relation

$$(x^2 + y^2)^2 = x^2 - y^2, \quad 0 \leq x \leq 1, \quad 0 \leq y. \quad (3)$$

Find an anti-derivative  $F(x)$  for  $f(x)$  with initial condition  $F(1) = 0$ .

**Analytic Solution:** Substituting  $x = \rho \cos \theta$  and  $y = \rho \sin \theta$  in relation (3), the Cartesian relation is converted into a polar relation as  $\rho^4 = \rho^2 \cos^2 \theta - \rho^2 \sin^2 \theta = \rho^2 \cos 2\theta$ , or  $\rho^2 = \cos 2\theta$ , with the following graph



Since from relation (2) we know that the graph goes through the point  $(1,0)$ , let  $F(x)$  be the antiderivative of  $f(x)$  defined by  $F(x) = A_x = \int_x^1 f(x) dx$ , so that  $F(x)$  would be the area under the curve directly above the interval  $[x, 1]$ , and  $F(1) = 0$ , and also let  $A_\theta = \frac{1}{2} \int_0^{\tan^{-1}[\frac{f(x)}{x}]} \rho(\theta)^2 d\theta$  be the polar area enclosed by the curve and the rays of lines  $\theta = 0$ ,  $\theta = \tan^{-1}[\frac{f(x)}{x}]$ . Then as seen in the diagram,

$$A_x = A_\theta - \frac{x f(x)}{2}.$$

Now, by the Fundamental Theorem of Calculus,

$$A_x = F(1) - F(x) = \frac{1}{2} \int_0^{\tan^{-1}[\frac{f(x)}{x}]} \cos 2\theta d\theta - \frac{x f(x)}{2}.$$

Hence, considering  $F(1) = \frac{x f(x)}{2} - \frac{1}{4} [\sin(2\theta)] \Big|_0^{\tan^{-1}[\frac{f(x)}{x}]}$

Or,

$$F(x) = \frac{x f(x)}{2} - \frac{x f(x)}{2[x^2 + f(x)^2]}.$$

Therefore, the above  $F(x)$  is the antiderivative of the implicitly defined function by relation (2) satisfying  $F(1) = 0$ .

**Direct Verification of the Solution:** Again, we will show the derivative of  $F(x)$  as expressed above is  $y = f(x)$ . To this end, again at some point in the process of the manipulation below, when the expression  $(x^2 + y^2)^2$  shows up as a denominator will make use of the implicit relation (2), and replace the expression by  $(x^2 - y^2)$ .

$$\begin{aligned} F'(x) &= \frac{d}{dx} \left[ \frac{xy}{2} - \frac{xy}{2(x^2 + y^2)} \right] = \\ &= \frac{y + xy'}{2} - \frac{1}{2} \frac{(y + xy')(x^2 + y^2) - xy(2x + 2yy')}{(x^2 + y^2)^2} = \\ &= \frac{y + xy'}{2} - \frac{xy'(x^2 - y^2) - y(x^2 - y^2)}{2(x^2 - y^2)} = \\ &= \frac{y + xy'}{2} - \frac{xy' - y}{2} = y. \end{aligned}$$

**Remark 1:** Incidentally the implicit relation (2) in Example 2, under the given conditions  $0 \leq x \leq 1$ ,  $0 \leq y$  can be solved for  $y$  in terms of  $x$  as the (non-handsome!) expression,

$$y = \frac{1}{\sqrt{2}} \sqrt{\sqrt{8x^2 + 1} - (2x^2 + 1)}, \quad 0 \leq x \leq 1.$$

Therefore, indeed the above solution to Example 2 has led to finding the solution to the following rather challenging integral,

$$\int \frac{1}{\sqrt{2}} \sqrt{\sqrt{8x^2 + 1} - (2x^2 + 1)} dx, \quad 0 \leq x \leq 1$$

as

$$\int \frac{1}{\sqrt{2}} \sqrt{\sqrt{8x^2 + 1} - (2x^2 + 1)} dx = \frac{xf(x)}{2} - \frac{xf(x)}{2[x^2 + f(x)^2]} + C,$$

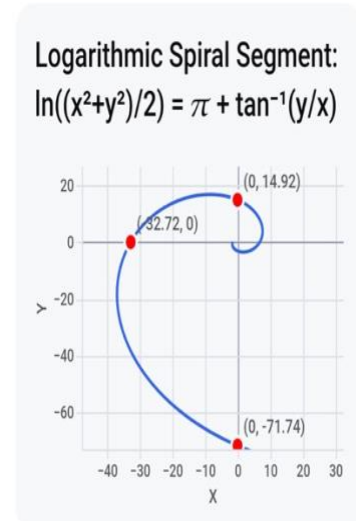
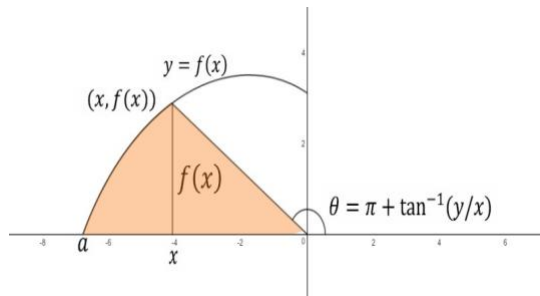
$$\text{where } f(x) = \frac{1}{\sqrt{2}} \sqrt{\sqrt{8x^2 + 1} - (2x^2 + 1)}, \quad 0 \leq x \leq 1.$$

This fact can be considered as a useful technique of integration to find specific class of integrals, as the author for one doesn't recall any integration technique that can integrate the above function where  $f(x)$ .

**Example 3** Find an antiderivative for the function Let  $y = f(x)$  implicitly defined by the relation

$$\ln\left(\frac{x^2 + y^2}{2}\right) = \pi + \tan^{-1}\left(\frac{y}{x}\right), \quad -\sqrt{2}e^{\frac{\pi}{2}} \leq x < 0, \quad 0 < y < \sqrt{2}e^{\frac{3\pi}{4}} \quad (4)$$

Note that under the restricted intervals for variables  $x$  and  $y$  in (3) the point  $(a, 0)$  with  $a = -\sqrt{2}e^{\frac{\pi}{2}}$  is the  $x$ -intercept of  $y = f(x)$ , whereas because of the fact that  $x = 0$  isn't in the domain of the function  $f(x)$  doesn't have a  $y$ -intercept.



**Analytic Solution:** Substituting  $x = \rho \cos\theta$  and  $y = \rho \sin\theta$  in relation (4), and considering the domain and range of the function, the polar angle between positive  $x$ -axis and the line segment from the origin to the point  $(x, f(x))$  will be represented by  $\theta = \pi + \tan^{-1}\left(\frac{y}{x}\right)$  with

$$\pi < \theta < \frac{3\pi}{2}, \text{ and the polar version of relation (3) would become } \ln\left(\frac{\rho^2}{2}\right) = \theta, \text{ or } \rho^2 = 2e^\theta.$$

Now let  $F(x)$  be the antiderivative defining the area under the curve that is directly above the interval  $[a, x]$ . Then, considering that the positive number  $-\frac{xf(x)}{2}$  represents the area of the

right triangle part of the shaded area in the figure above, the total shaded polar area  $A_\theta$  will be the same as

$$\frac{1}{2} \int_0^{\tan^{-1}[\frac{f(x)}{x}]} \rho(\theta)^2 d\theta = F(x) - \frac{x f(x)}{2} = \frac{1}{2} \int_0^{\tan^{-1}[\frac{f(x)}{x}]} 2e^\theta d\theta = e^\theta \Big|_0^{\tan^{-1}[\frac{f(x)}{x}]} = e^{\tan^{-1}[\frac{f(x)}{x}]} - e^0 = e^{\tan^{-1}[\frac{f(x)}{x}]} - 1.$$

Therefore 
$$F(x) = \frac{xy}{2} - e^{\pi + \tan^{-1}[\frac{y}{x}]} + e^\pi$$

is the required antiderivative of the implicitly defined function  $y = f(x)$  by relation (3), under the given restrictions of the variables.

**Direct Verification of the Solution:**

$$F'(x) = \frac{d}{dx} \left[ \frac{xy}{2} - e^{\pi + \tan^{-1}[\frac{y}{x}]} \right] = \frac{1}{2} [f(x) + x f'(x)] - \frac{x f'(x) - f(x)}{[x^2 + f(x)^2]} e^{\pi + \tan^{-1}[\frac{f(x)}{x}]}.$$

Now, since relation (3) implies  $[x^2 + f(x)^2] = 2e^{\pi + \tan^{-1}[\frac{f(x)}{x}]}$ , we conclude that  $F'(x) = \frac{1}{2} [f(x) + x f'(x)] - \frac{1}{2} [x f'(x) - f(x)] = f(x)$ , and verification of the solution is complete.

**Remark 2:** Since the graph on the right also is represented by the same relation  $R(x, y) = 0$  and this particular extension defines three distinct functions defined on the appropriate intervals, the direct verification above shows that all three functions have the same general antiderivative

$$F(x) = \frac{xy}{2} - e^{\pi + \tan^{-1}[\frac{y}{x}]} + e^\pi + C$$

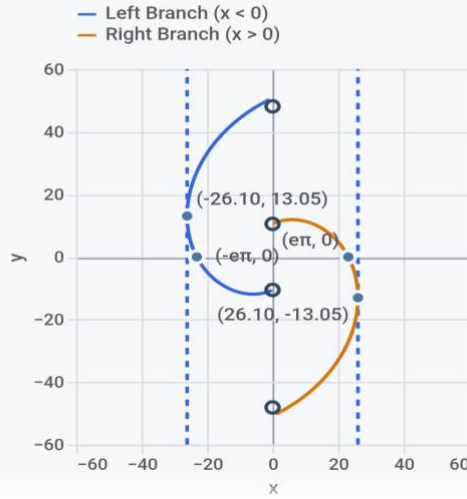
Over the same appropriate intervals.

**Example 4** Find all antiderivatives for the four distinct implicitly defined functions  $y = f(x)$  by the relation

$$\ln(x^2 + y^2) = 2\pi - \tan^{-1}\left(\frac{y}{x}\right), \quad x \neq 0, \quad \theta = \tan^{-1}\left(\frac{y}{x}\right) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \cup (0, \pi) \quad (5)$$

with the following graph,

Equation:  $\ln(x^2 + y^2) = 2\pi - \tan^{-1}(y/x)$



The relation (5) defines four distinct implicitly defined functions, simply because (thanks to AI assistance now available on Google!) the vertical lines  $x = -\frac{e^{\pi + \frac{1}{2}\tan^{-1}(\frac{1}{2})}}{\sqrt{5}} \cong -26.1$  and  $x = \frac{e^{\pi + \frac{1}{2}\tan^{-1}(\frac{1}{2})}}{\sqrt{5}} \cong 26.1$  are the only two possible vertical tangent lines to the graph of the relation

(5). So that there are two distinct functions defined on open interval  $I_1 = (0, \frac{e^{\pi + \frac{1}{2}\tan^{-1}(\frac{1}{2})}}{\sqrt{5}}]$ , one whose graph entirely lies in quadrant IV, and we will denote by  $y = f_{IV}(x)$ , and the other one  $y = f_I(x)$  which have most of its as its graph in quadrant I, while all of it is directly above graph of  $f_{IV}(x)$ . Similarly, there are two other distinct functions  $y = f_{II}(x)$  and  $y = f_{III}(x)$  defined on open interval  $I_2 = [-\frac{e^{\pi + \frac{1}{2}\tan^{-1}(\frac{1}{2})}}{\sqrt{5}}, 0)$ .

Note that the domain intervals are half open because  $x \neq 0$ , so that the graph of doesn't have any  $y$ -intercepts, however there are two  $x$ -intercepts which are  $(e^\pi, 0) \in I_1$  and  $(-e^\pi, 0) \in I_2$ . While, similar to the previous Direct Verification arguments, the reader can verify the

Function  $F(x) = \frac{xy}{2} + \frac{1}{2}[e^{2\pi - \tan^{-1}(\frac{y}{x})}]$  is an antiderivative for all four distinct functions, in the following we can still use the method of the Theorem to find particular antiderivatives for each of them separately.

We will first find an antiderivative for  $y = f_I(x)$  whose domain contain  $(e^\pi, 0)$ . To this end we choose the point  $a$  in the Theorem to be  $a = e^\pi$ , and (for more clarity of visualising in the above figure) we assume  $x < e^\pi$ . Next

we let  $F_I(x)$  be the antiderivative for  $y = f_I(x)$  that (by fundamental theorem of calculus) defines the area above the interval  $[x, e^\pi]$  as  $F_I(e^\pi) - F_I(x)$ . Then, since the polar version of the relation (5) is  $\rho^2 = e^{2\pi - \theta}$ , we have

$$F_I(x) + \frac{xy}{2} = \frac{1}{2} \int_0^{\tan^{-1}[\frac{y}{x}]} e^{2\pi-\theta} d\theta = \frac{xy}{2} - \frac{1}{2} e^{2\pi-\theta} \Big|_0^{\tan^{-1}[\frac{y}{x}]} = \frac{xy}{2} - \frac{1}{2} [e^{2\pi-\tan^{-1}[\frac{y}{x}]} - e^{2\pi}].$$

Therefore, since  $F_I(e^\pi) = 0$ , we get

$$F_I(x) = \frac{xy}{2} + \frac{1}{2} [e^{2\pi-\tan^{-1}[\frac{y}{x}]} - e^{2\pi}].$$

and

$$F_I(x) = \frac{xy}{2} + \frac{1}{2} [e^{2\pi-\tan^{-1}(\frac{y}{x})} - e^{2\pi}]$$

Is required antiderivative for  $y = f_I(x)$  with initial condition  $F(e^\pi) = 0$ .

Next since the graph of the relation (5) is symmetric through the origin, we can easily find the antiderivative for  $y = f_{III}(x)$  by considering that  $f_{III}(x) = -f_I(-x)$ ,

$$\begin{aligned} F_{III}(x) &= \int f_{III}(x) dx = \int -f_I(-x) dx = \int f_I(u) du = F_I(u) = \frac{u f_I(u)}{2} + \frac{1}{2} [e^{2\pi-\tan^{-1}[\frac{f_I(u)}{u}]} - e^{2\pi}] \\ &= \frac{-x f_I(-x)}{2} + \frac{1}{2} [e^{2\pi-\tan^{-1}[\frac{f_I(-x)}{-x}]} - e^{2\pi}] = \frac{-x [-f_{III}(x)]}{2} + \frac{1}{2} [e^{2\pi-\tan^{-1}[\frac{-f_{III}(x)}{-x}]} - e^{2\pi}] = \\ &= \frac{x f_{III}(x)}{2} + \frac{1}{2} [e^{2\pi-\tan^{-1}[\frac{f_{III}(x)}{x}]} - e^{2\pi}]. \text{ Or, } F_{III}(x) = \frac{xy}{2} + \frac{1}{2} [e^{2\pi-\tan^{-1}[\frac{y}{x}]} - e^{2\pi}], \text{ meaning that} \end{aligned}$$

the same antiderivative for  $f_I(x)$  acts as an antiderivative for  $f_{III}(x)$ , however on (opposite) domain and with the initial condition  $F_{III}(-e^\pi) = 0$ .

Next, we will find an antiderivative for  $f_{II}(x)$ . To this end, since the point of intersection of the graph with the line  $y = x$  in quadrant II is  $(-\frac{e^{\frac{9\pi}{8}}}{2}, \frac{e^{\frac{9\pi}{8}}}{2})$ , considering  $a = -\frac{e^{\frac{9\pi}{8}}}{2} \in I_2$ , the

$$\begin{aligned} F_{II}(x) &= \frac{xy}{2} - \frac{1}{2} \int_{\frac{3\pi}{4}}^{\tan^{-1}(\frac{f(x)}{x})} \rho(\theta)^2 d\theta - \frac{af(a)}{2} = \frac{xy}{2} + \frac{1}{2} e^{2\pi-\theta} \Big|_{\frac{3\pi}{4}}^{\tan^{-1}[\frac{y}{x}]} = \frac{xy}{2} + \\ &[e^{2\pi-\tan^{-1}[\frac{y}{x}]} - \frac{e^{\frac{5\pi}{4}}}{2} + \frac{e^{\frac{9\pi}{4}}}{8}] \end{aligned}$$

$$\text{Or } F_{II}(x) = \frac{xy}{2} + \frac{1}{2} e^{2\pi-\tan^{-1}(\frac{y}{x})} + \frac{e^{\frac{9\pi}{4}}}{8},$$

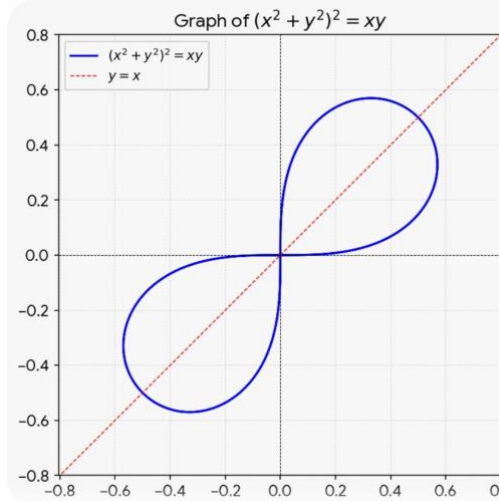
with the initial condition  $F_{II}(0) = \frac{e^{\frac{5\pi}{4}}}{2}$ .

It should now be easy for reader to use the same argument we used to find the particular antiderivative for  $f_{III}(x)$  from that of  $f_I(x)$  earlie, and find a particular antiderivative for  $f_{IV}(x)$  from the antiderivative of  $y = f_{II}(x)$ , as the two functions are symmetric through the origin.

**Remark 3:** What happen in the above Example 4, that all implicitly defined functions defined by a relation can have the same universal antiderivative can't be generalised. For example, in the case of the unit circle defined by  $x^2 + y^2 = 1$ , the two implicitly defined functions are  $y = \sqrt{1-x^2}$  and  $y = -\sqrt{1-x^2}$  which obviously have different antiderivatives (Application 1 on page 17 shows how our method finds the antiderivative for  $y = \sqrt{1-x^2}$ , and the antiderivative for  $y = -\sqrt{1-x^2}$  will be negative that).

**Example 5** Find an anti-derivative  $F(x)$  for the function  $y = f(x)$  implicitly defined by the relation

$$(x^2 + y^2)^2 = xy, \quad 0 < x < \frac{\sqrt[4]{27}}{4}, \quad 0 < y < \frac{\sqrt[4]{27}}{4}. \quad (6)$$



Note, that the above graph is an Example of a Lemniscate of Bernoulli.

**Analytic Solution:** Substituting  $x = \rho \cos \theta$  and  $y = \rho \sin \theta$  in relation (6), the Cartesian relation is converted into a polar relation as  $\rho^4 = \rho^2 \sin \theta \cos \theta$ , or  $\rho^2 = \frac{1}{2} \sin 2\theta$ , which also implies  $\rho = \sqrt{\frac{1}{2} \sin 2\theta}$ . Since graph of the relation (5) passes through the point  $(\frac{1}{2}, \frac{1}{2})$  we will choose the real number in the Theorem to be  $a = \frac{1}{2}$ , then the antiderivative  $F(x)$  for  $y = f(x)$  with  $F(\frac{1}{2}) = 0$  will be

$$F(x) = \frac{xy}{2} - \frac{1}{2} \int_0^{\tan^{-1}(\frac{y}{x})} \rho^2(\theta) d\theta - \frac{1}{8}$$

Therefore,

$$\begin{aligned} F(x) &= \frac{xy}{2} - \frac{1}{2} \int_{\frac{\pi}{4}}^{\tan^{-1}(\frac{y}{x})} \frac{1}{2} \sin 2\theta d\theta - \frac{1}{8} = \frac{xy}{2} - \frac{1}{2} \int_{\frac{\pi}{4}}^{\tan^{-1}(\frac{y}{x})} \sin \theta \cos \theta d\theta - \\ & - \frac{1}{8} = \frac{xy}{2} + \left[ \frac{1}{4} \sin^2 \theta \right]_{\tan^{-1}(\frac{y}{x})}^{\frac{\pi}{4}} - \frac{1}{8} \\ &= \frac{xy}{2} + \frac{y^2}{4(x^2 + y^2)} - \frac{1}{8}, \end{aligned}$$

and hence

$$F(x) = \frac{xy}{2} + \frac{y^2}{4(x^2 + y^2)} - \frac{1}{8}$$

is the required antiderivative for the implicitly defined function by relation (6).

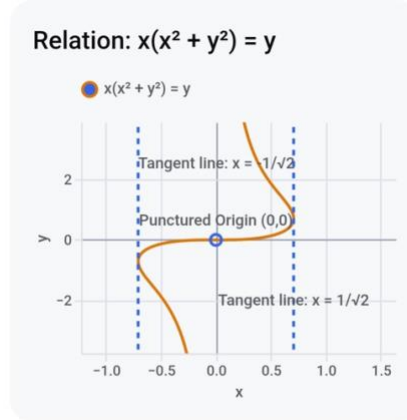
**Direct Verification of the Solution:** We will show the derivative of  $F(x)$  as expressed above is  $y = f(x)$ . And again, to this end, in the following we make use of relation (6) by substituting  $(x^2 + y^2)^2$  with  $xy$  at some point.

$$\begin{aligned} F'(x) &= \frac{d}{dx} \left[ \frac{xy}{2} + \frac{x^2}{4(x^2+y^2)} - \frac{1}{8} \right] = \\ F'(x) &= \frac{y+xy'}{2} + \frac{1}{4} \frac{2x(x^2+y^2) - x^2(2x+2yy')}{(x^2+y^2)^2} = \\ &= \frac{y+xy'}{2} + \frac{2xy^2 - 2x^2yy'}{4xy} = \frac{y+xy'}{2} + \frac{y-xy'}{2} = y. \end{aligned}$$

Therefore  $F'(x) = y = f(x)$ , and the verification is established.

**Example 6** Let  $y = f(x)$  be the function implicitly defined by the following relation whose graph is the serpentine shown below,

$$x(x^2 + y^2) = y, \quad 0 \leq |x| < \frac{1}{\sqrt{2}}, \quad 0 \leq |y| < \frac{1}{\sqrt{2}}. \quad (7)$$



Find an anti-derivative  $F(x)$  for  $f(x)$ .

**Analytic Solution:** Substituting  $x = \rho \cos \theta$  and  $y = \rho \sin \theta$  in relation (7), the Cartesian relation is converted into a polar relation as  $\rho^2 = \tan \theta$ .

As the vertical tangent lines  $x = -\frac{1}{\sqrt{2}}$  and  $x = \frac{1}{\sqrt{2}}$  divide the graph into three functions.

First we will find an antiderivative for the implicitly defined function with domain  $\left(0, \frac{1}{\sqrt{2}}\right]$  whose graph is above the horizontal line  $y = \frac{1}{\sqrt{2}}$ , and has the  $y$ -axis as an asymptote. Since the point  $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$  is on its graph, we will choose the real number in the Theorem to be  $a = \frac{1}{\sqrt{2}}$ , then the antiderivative  $F(x)$  for  $y = f(x)$  with  $F\left(\frac{1}{\sqrt{2}}\right) = 0$  will be

$$F(x) = \frac{xy}{2} - \frac{1}{2} \int_0^{\tan^{-1}\left(\frac{y}{x}\right)} \rho^2(\theta) d\theta - \frac{1}{4}$$

Hence the antiderivative will be obtained as follows,

$$F(x) = \frac{x f(x)}{2} - \frac{1}{2} \int_0^{\tan^{-1}[\frac{f(x)}{x}]} \tan \theta \, d\theta - \frac{1}{4} = \frac{x f(x)}{2} + \frac{1}{2} \ln(\cos \theta)$$

$$\left| \tan^{-1}[\frac{f(x)}{x}] - \frac{1}{4} \right| = \frac{x f(x)}{2} + \frac{1}{2} \ln\left(\frac{x}{\sqrt{x^2+y^2}}\right)$$

and therefore

$$F(x) = \frac{x f(x)}{2} + \frac{1}{2} \ln\left(\frac{x}{\sqrt{x^2+y^2}}\right) - \frac{1}{4}$$

is the required antiderivative for of the function  $y = f(x)$  implicitly defined by relation (7).

**Verification of the Solution:** To show that the derivative of  $F(x)$  as expressed above is  $y = f(x)$ , as an easy and routine exercise to the reader to use rules of differentiation to show  $\frac{d}{dx} \left[ \ln\left(\frac{x}{\sqrt{x^2+y^2}}\right) - \frac{1}{4} \right] = \frac{y^2 - xyy'}{x(x^2+y^2)}$ . Having shown this, if we replace the denominator

$x(x^2+y^2)$  by  $y$  from relation (5), we get  $\frac{d}{dx} \left[ \ln\left(\frac{x}{\sqrt{x^2+y^2}}\right) \right] = \frac{y^2 - xyy'}{y} = \frac{y+xy'}{2}$ . Therefore,  
 $F'(x) = \frac{d}{dx} \left[ \frac{xy}{2} + \frac{1}{2} \ln\left(\frac{x}{\sqrt{x^2+y^2}}\right) - \frac{1}{4} \right] = \frac{y+xy'}{2} + \frac{y+xy'}{2} = y$ , or  $F'(x) = y = f(x)$ , meaning  $F(x)$  is indeed an antiderivative of  $f(x)$ , and the verification is complete.

Note that the above verification is also a proof that

$$F(x) = \frac{x f(x)}{2} + \frac{1}{2} \ln\left(\frac{x}{\sqrt{x^2+y^2}}\right) + C$$

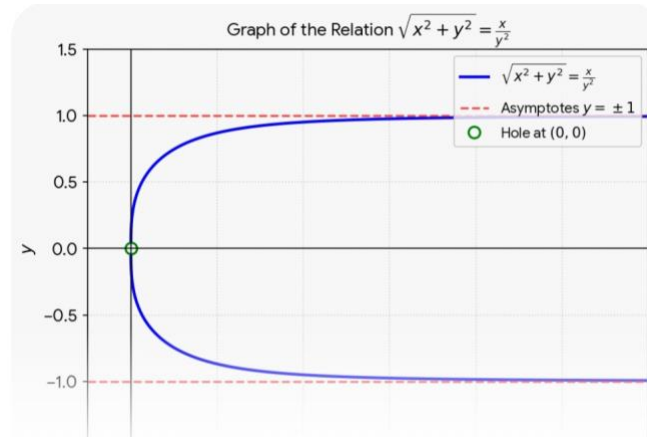
Is the general antiderivative for the implicitly defined function whose graph is the left part of the graph of the original cartesian relation, that is for  $h(x) = -f(-x)$ ,  $x \in [-\frac{1}{\sqrt{2}}, 0)$  whose graph again has the  $y$ -axis as an asymptote. Also note that for the third implicitly defined function with domain  $[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]$ , the above general antiderivative will not work [because  $\ln\left(\frac{x}{\sqrt{x^2+y^2}}\right)$  isn't define for  $x = y = 0$ ], however if we exclude the origin and break this function into two separate functions with domains  $[-\frac{1}{\sqrt{2}}, 0)$  and  $(0, \frac{1}{\sqrt{2}}]$ , the above verification shows that

$$F(x) = \frac{x f(x)}{2} + \frac{1}{2} \ln\left(\frac{x}{\sqrt{x^2+y^2}}\right) + C$$

Is a general antiderivative for all the four distinct implicitly defined functions by relation (7).

**Example 7** Let  $y = f(x)$  be the function implicitly defined by the relation

$$\sqrt{x^2 + y^2} = \frac{x}{y^2}, \quad 0 < x. \quad (8)$$



Find an anti-derivative  $F(x)$  for  $f(x)$ .

**Analytic Solution:** Substituting  $x = \rho \cos \theta$  and  $y = \rho \sin \theta$  in relation (8), the Cartesian relation is converted into a polar relation as  $\rho^2 = \frac{\cos \theta}{\sin^2 \theta}$ .

Considering that the point with the Cartesian coordinates  $(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{2}})$  is on the graph of relation (8), to find an anti-derivative for  $f(x)$  it is enough to use the version

$$F(x) = \frac{x f(x)}{2} - \frac{1}{2} \int_{\tan^{-1}[\frac{\sqrt{2}}{\sqrt{6}}]}^{\tan^{-1}[\frac{f(x)}{x}]} \rho^2(\theta) d\theta$$

of the Theorem. Hence the antiderivative will be obtained as follows,

$$F(x) = \frac{x f(x)}{2} - \frac{1}{2} \int_{\tan^{-1}[\frac{\sqrt{2}}{\sqrt{6}}]}^{\tan^{-1}[\frac{f(x)}{x}]} \frac{\cos \theta}{\sin^2 \theta} d\theta =$$

$$\frac{x f(x)}{2} + \frac{1}{2 \sin \theta} \Bigg|_{\tan^{-1}[\frac{\sqrt{2}}{\sqrt{6}}]}^{\tan^{-1}[\frac{f(x)}{x}]} = \frac{x f(x)}{2} + \frac{\sqrt{x^2 + y^2}}{2y} - \frac{\pi}{6}$$

and therefore

$$F(x) = \frac{x y}{2} + \frac{\sqrt{x^2 + y^2}}{2y} - \frac{\pi}{6}$$

is the antiderivative of the function  $y = f(x)$  implicitly defined by relation (7), with  $F(\frac{1}{6}) = 0$ .

**Direct verification of the Solution:**

$$F'(x) = \frac{d}{dx} \left[ \frac{xy}{2} + \frac{\sqrt{x^2 + y^2}}{2y} - \frac{\pi}{6} \right] =$$

$$\begin{aligned}
& \frac{1}{2} \left[ y + xy' + \frac{\frac{y(x + yy')}{\sqrt{x^2 + y^2}} - \sqrt{x^2 + y^2} y'}{y^2} \right] = \\
& \frac{1}{2} \left[ y + xy' + \frac{y(x + yy') - (x^2 + y^2)y'}{y^2 \sqrt{x^2 + y^2}} \right] = \\
& = \frac{1}{2} \left[ y + xy' + \frac{y(x + yy') - (x^2 + y^2)y'}{x} \right],
\end{aligned}$$

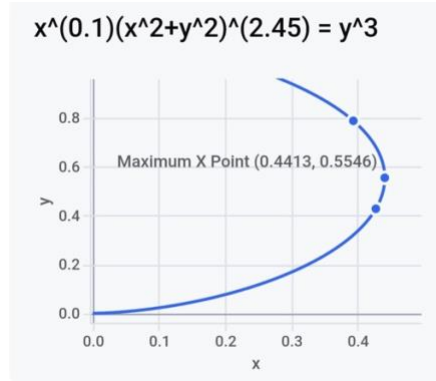
where the last step is implied by relation (7). Therefore

$$\begin{aligned}
F'(x) &= \frac{1}{2} \left[ y + xy' + \frac{xy + y^2 y' - x^2 y' - y^2 y'}{x} \right] = \\
&= \frac{1}{2} \left[ y + xy' + \frac{x(y - xy')}{x} \right] = y = f(x).
\end{aligned}$$

Hence the verification is complete.

**Example 8** Let  $y = f(x)$  be the implicitly defined function as follows, and with the following graph,

$$x^{0.1} (x^2 + y^2)^{4.5} = y^3, \quad 0 \leq x. \quad (9)$$



Find an anti-derivative  $F(x)$  for  $f(x)$ .

**Solution:** It is easy to see that the polar version of relation (9) reduces to  $\rho^2 = \sin^3 \cos^{0.1} \theta$ . Or,

$\rho^2 = \sin^3 \theta \cos^{0.1} \theta = \sin \theta (1 - \cos^2 \theta) \cos^{0.1} \theta = \sin \theta \cos^{0.1} \theta - \sin \theta \cos^{2.1} \theta$ , and that

$$\int \rho^2 (\theta) d\theta = \int (\sin \theta \cos^{0.1} \theta - \sin \theta \cos^{2.1} \theta) d\theta =$$

$-\frac{10}{9}\cos^{0.9}\theta + \frac{10}{29}\cos^{2.9}\theta$ . Since  $f(0) = 0$ , as for the lower branch function (to the maximum  $x$  shown above),

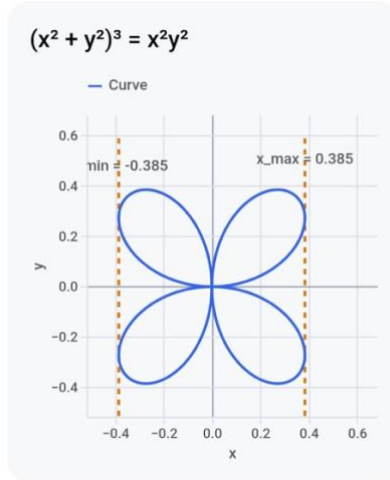
$$\begin{aligned} F(x) &= \frac{x f(x)}{2} - \frac{1}{2} \int_0^{\tan^{-1}[\frac{f(x)}{x}]} \rho^2(\theta) d\theta = \\ F(x) &= \frac{x f(x)}{2} - \frac{1}{2} \left( -\frac{10}{9}\cos^{0.9}\theta + \frac{10}{29}\cos^{2.9}\theta \right), \text{ or} \\ F(x) &= \frac{x f(x)}{2} + \frac{5}{9} \frac{\sqrt[10]{x^9}}{\sqrt[20]{(x^2 + y^2)^9}} - \frac{5}{29} \frac{\sqrt[10]{x^{29}}}{\sqrt[20]{(x^2 + y^2)^{29}}} \end{aligned}$$

Serves as an antiderivative for the two distinct functions representing both lower and upper graphs the maximum  $x$  shown in the figure.

Note that, it can be shown that the lower graph is tangent to the  $x$ -axis from the right side, and the upper graph has the  $y$ -axis as its asymptote.

**Example 9** Consider the following relation, whose graph is the quadrifolium bellow it.

$$(x^2 + y^2)^3 = x^2 y^2, \quad x, y \in I = \left[ -\frac{2\sqrt{3}}{9}, \frac{2\sqrt{3}}{9} \right]. \quad (10)$$



The vertical tangent lines  $x = -\frac{2\sqrt{3}}{9}$  and  $x = \frac{2\sqrt{3}}{9}$  divide the entire graph into four distinct implicitly defined functions all with the same domain  $I = \left[ -\frac{2\sqrt{3}}{9}, \frac{2\sqrt{3}}{9} \right]$ . Graphs of two of the functions look like semi-ellipses; one is concave up in quadrants I and II, and the other one's graph is just the reflection of the first one through  $x$ -axis. The third function's graph resembles a vertically stretched S in quadrants I and IV tangent to the  $x$ -axis at the origin, and the graph of the fourth one is just the reflection of the latter graph through the  $y$ -axis. Because of all this symmetry, and since all four distinct functions have graphs tangent to the  $x$ -axis at the origin we can choose  $a = 0$  in the Theorem, and considering that the polar representation of relation (10) simplifies into

$\rho^2 = \sin^2\theta \cos^2\theta$ , we get the following Universal antiderivative for all the four distinct functions  $y = f(x)$  as follows,

$$\begin{aligned}
F(x) &= \frac{x f(x)}{2} - \frac{1}{2} \left[ \int_0^{\tan^{-1}\left(\frac{y}{x}\right)} \sin^2 \theta \cos^2 \theta \, d\theta \right] = \frac{x f(x)}{2} - \frac{1}{2} \left[ \frac{\theta}{8} - \frac{\sin 4\theta}{32} \right] + C = \\
&\quad \frac{x y}{2} + \frac{1}{2} \left[ \frac{2 \sin 2\theta \cos 2\theta}{32} - \tan^{-1}\left(\frac{y}{x}\right) \right] + C = \\
&\quad \frac{x y}{2} + \frac{1}{2} \left[ \frac{\sin \theta \cos \theta (2 \cos^2 \theta - 1)}{8} - \tan^{-1}\left(\frac{y}{x}\right) \right] + C = \\
&\quad \frac{x y}{2} + \frac{1}{2} \left[ \frac{2 x^3 y}{8 (x^2 + y^2)^2} - \frac{x y}{8 (x^2 + y^2)} - \frac{1}{8} \tan^{-1}\left(\frac{y}{x}\right) \right] + C = \\
&\quad \frac{x y}{2} + \frac{1}{16} \left[ \frac{2 x^3 y}{(x^2 + y^2)^2} - \frac{x y}{(x^2 + y^2)} - \tan^{-1}\left(\frac{y}{x}\right) \right] + C.
\end{aligned}$$

Hence,

$$F(x) = \frac{1}{16} \left[ 8x y + \frac{2x^3 y}{(x^2 + y^2)^2} - \frac{x y}{(x^2 + y^2)} - \tan^{-1}\left(\frac{y}{x}\right) \right] + C$$

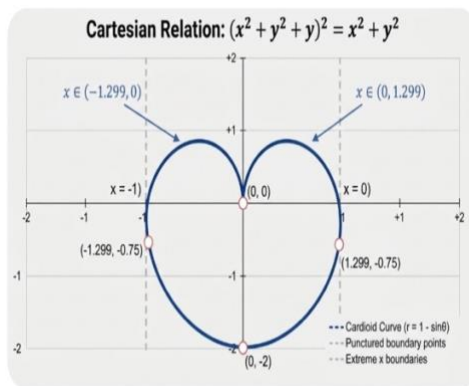
will be the universal antiderivative for all the four distinct functions  $y = f(x)$  defined by relation (10). However, if you multiply the top if you write the first fraction inside the bracket as

$\frac{2 x^3 y}{(x^2 + y^2)^2} = \frac{2 x^3 y (x^2 + y^2)^2}{(x^2 + y^2)^3}$ , and the second one as  $\frac{x y}{(x^2 + y^2)} = \frac{x y (x^2 + y^2)^2}{(x^2 + y^2)^3}$  and make use of the relation (10) twice, and take a common denominator from the first three terms of the bracket, after some simplification the above universal antiderivative can be expressed easier as

$$F(x) = \frac{1}{16} \left[ \frac{x^4 + 8x^2 y^2 - y^4}{x y} - \tan^{-1}\left(\frac{y}{x}\right) \right] + C$$

**Example 10** Consider the following relation, with a Cardioid graph.

$$(x^2 + y^2)^3 = x^2 y^2, \quad x, y \in I = \left[ -\frac{2\sqrt{3}}{9}, \frac{2\sqrt{3}}{9} \right]. \quad (11)$$



As shown in the above graph, the vertical tangent lines  $x = -\frac{3\sqrt{3}}{4}$  and  $x = \frac{3\sqrt{3}}{4}$  divide the entire graph into four functions, however as we will see shortly the antiderivative of any of the two functions will involve the term  $\tan^{-1}\left(\frac{y}{x}\right)$ , which isn't defined at  $y$ -intercepts  $(0, 0)$  and  $(0, -2)$ . So as far as finding antiderivatives is involved, the best is to break up the cardioid graph (with  $y$ -intercepts removed) into four implicitly defined integrable functions, two of them with domain  $[-\frac{3\sqrt{3}}{4}, 0)$  with graphs up and down the horizontal line  $y = -0.75 = -\frac{1}{4}$ , and the other two with domain  $(0, \frac{3\sqrt{3}}{4}]$  and again with graphs up and down the line  $y = -\frac{1}{4}$ . Let us first use the Theorem and find a particular antiderivative for the function with domain  $(0, \frac{3\sqrt{3}}{4}]$  whose graph is above the line  $y = -\frac{1}{4}$ . To this end, since (as the reader can verify) the polar version of relation (11) reduces to  $\rho = 1 - \sin \theta$ , we will have,

$$\rho^2 = (1 - \sin \theta)^2 = 1 - 2 \sin \theta + \sin^2 \theta = 1 - 2 \sin \theta + \frac{1 - \cos 2 \theta}{2} = \frac{3}{2} - 2 \sin \theta - \frac{1}{2} \cos 2 \theta.$$

Hence, considering that  $(1, 0)$  is a point on the graph of the function under consideration, we will choose  $a = 1$ . Then the Theorem provides the following antiderivative,

$$\begin{aligned} F(x) &= \frac{xy}{2} - \frac{1}{2} \left[ \int_0^{\tan^{-1}(\frac{y}{x})} \left( \frac{3}{2} - 2 \sin \theta - \frac{1}{2} \cos 2 \theta \right) d\theta \right] = \\ &= \frac{xy}{2} - \frac{1}{2} \left[ \frac{3}{2} \theta + 2 \cos \theta - \frac{1}{2} \sin 2 \theta \right]_{\theta=0}^{\theta=\tan^{-1}(\frac{y}{x})} = \\ &= \frac{xy}{2} - \frac{1}{2} \left[ \frac{3}{2} \tan^{-1}\left(\frac{y}{x}\right) + \frac{2x}{\sqrt{x^2 + y^2}} - \frac{xy}{2(x^2 + y^2)} - 2 \right] = \\ &= \frac{xy}{2} - \frac{3}{4} \tan^{-1}\left(\frac{y}{x}\right) - \frac{x}{\sqrt{x^2 + y^2}} + \frac{xy}{4(x^2 + y^2)} + 2 \end{aligned}$$

Hence,

$$F(x) = \frac{xy}{2} - \frac{x}{\sqrt{x^2 + y^2}} + \frac{xy}{4(x^2 + y^2)} - \frac{3}{4} \tan^{-1}\left(\frac{y}{x}\right) + 2$$

is a particular antiderivative for the function.

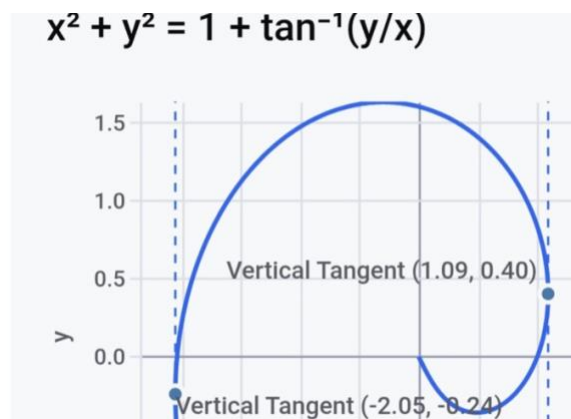
Indeed, as we have seen how enough verifications have been done so far, the reader should be able to use the rules of differentiation and show that  $\frac{dF(x)}{xd} = y$ . And this verification is enough to conclude that (dropping the constant 2) the function

$$F(x) = \frac{xy}{2} - \frac{x}{\sqrt{x^2 + y^2}} + \frac{xy}{4(x^2 + y^2)} - \frac{3}{4} \tan^{-1}\left(\frac{y}{x}\right)$$

is a universal antiderivative for all the four implicitly defined by relation (11), with restriction  $x \neq 0$ .

**Example 11** Consider the following relation, with the graph seen bellow,

$$x^2 + y^2 = 1 + \tan^{-1}\left(\frac{y}{x}\right), \quad -2.05 \leq x \leq 1.09, \quad x \neq 0 \quad (12)$$



Note that the numbers  $-2.05$  and  $1.09$  seen on the double inequality in (12) are the two transcendental numbers representing the abscissa of contact points of vertical tangent lines to the graph, rounded to two decimals, so that the equations  $x \approx -2.05$  and  $x \approx 1.09$  will represent the two vertical tangent lines to the graph. As seen in the figure, considering the restriction  $x \neq 0$  in (12), the mentioned vertical tangent lines divide the entire graph into three functions (rather than two) distinct functions. One with the domain  $-2.05 \leq x < 0$  and the other two both with domain  $0 < x \leq 1.09$ , the graph of one of which is entirely in quadrant I. We will first apply the Theorem and provide a particular antiderivative for the function with domain  $0 < x \leq 1.09$ , whose  $x$ -intercept happens to be at  $(1, 0)$ . So that we can choose  $a = 1$  in the Theorem. Since obviously the polar representation of relation (12) is  $\rho^2 = 1 + \theta$ , the Theorem implies

$$F(x) = \frac{xy}{2} - \frac{1}{2} \left[ \int_0^{\tan^{-1}(\frac{y}{x})} (1 + \theta) d\theta \right] =$$

$$\frac{xy}{2} - \frac{1}{2} \theta - \frac{1}{4} \theta^2 = \frac{xy}{2} - \frac{1}{2} \tan^{-1}\left(\frac{y}{x}\right) - \frac{1}{4} \left[ \tan^{-1}\left(\frac{y}{x}\right) \right]^2.$$

Hence,

$$F(x) = \frac{xy}{2} - \frac{1}{2} \tan^{-1}\left(\frac{y}{x}\right) - \frac{1}{4} \left[ \tan^{-1}\left(\frac{y}{x}\right) \right]^2$$

is a particular antiderivative for the chosen function.

Indeed, since the function with the domain  $-2.05 \leq x < 0$  also has an  $x$ -intercept at  $(-1, 0)$  it follows that the above is also a particular antiderivative for this function as well. It is now left to the reader to show that

$$F(x) = \frac{xy}{2} - \frac{1}{2} \tan^{-1}\left(\frac{y}{x}\right) - \frac{1}{4} \left[ \tan^{-1}\left(\frac{y}{x}\right) \right]^2 + C$$

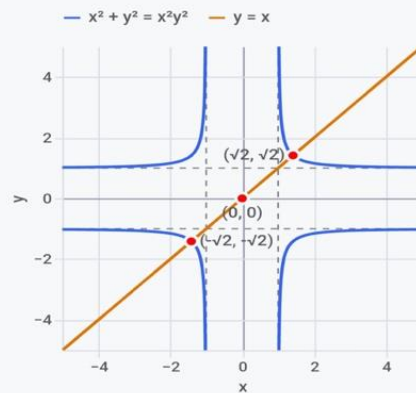
is the universal general antiderivative for all three functions, and to find a particular antiderivative for third function (if the particular antiderivative is to contain an exact value for its constant this could be challenging).

**Exercise 1** Let  $y = f(x)$  be the function implicitly defined by the relation

$$(x^2 + y^2) = x^2 y^2, \quad 1 < |x|, \quad 0 < |y| \quad (13)$$

With the following graph,

### Intersection of $x^2 + y^2 = x^2y^2$ and $y = x$



Find particular antiderivatives for all four distinct implicitly defined functions by relation (13), and conclude that

$$F(x) = \frac{1}{2} \left[ xy + \frac{x}{y} - \frac{y}{x} \right],$$

Is a universal antiderivative for all of them. More precisely, show that,

$$F_I(x) = \frac{1}{2} \left[ xy + \frac{x}{y} - \frac{y}{x} \right] - 1, \text{ with initial condition } F(\sqrt{2}) = 0 \text{ for the function quadrants I.}$$

$$F(x) = \frac{1}{2} \left[ xy + \frac{x}{y} - \frac{y}{x} \right] + 1, \text{ with initial condition } F(-\sqrt{2}) = 0 \text{ for both the pieces on quadrants II and III.}$$

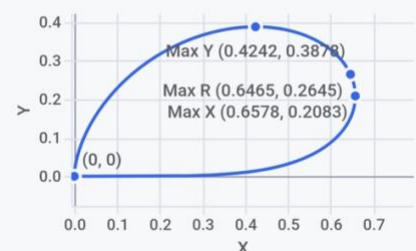
$$F(x) = \frac{1}{2} \left[ xy + \frac{x}{y} - \frac{y}{x} \right] + 1, \text{ with initial condition } F(-\sqrt{2}) = 0 \text{ for the piece on quadrants IV.}$$

**Exercise 2** Let  $y = f(x)$  be the function implicitly defined by the relation

$$y^{\frac{1}{2}} (x^2 + y^2)^{\frac{11}{4}} = x^3 \quad (14)$$

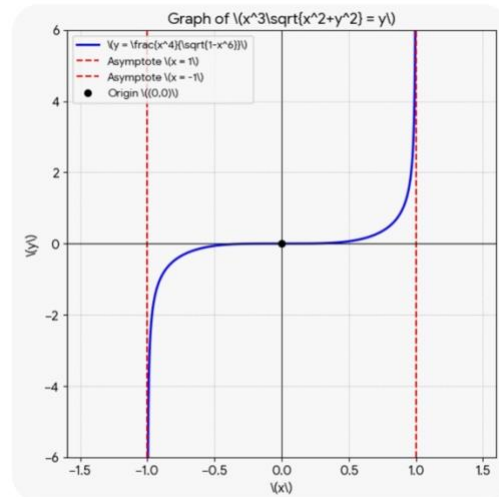
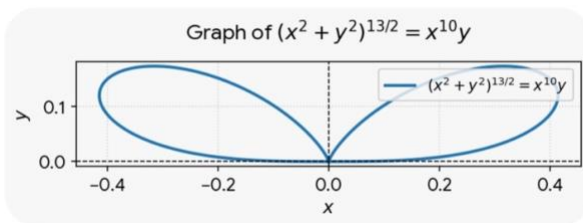
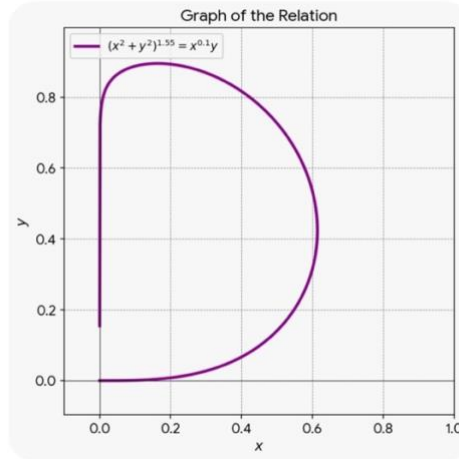
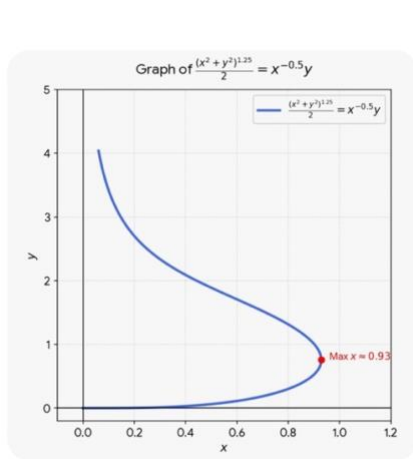
With the following teardrop graph,

### Graph of $y^{(-0.5)}(x^2 + y^2)^{(2.75)} = x^3$



Show that the function  $F(x) = \frac{xy}{2} - \frac{y^{\frac{3}{2}}}{3(x^2+y^2)^{\frac{3}{4}}} + \frac{y^{\frac{7}{2}}}{7(x^2+y^2)^{\frac{7}{4}}}$  is an antiderivative for both functions implicitly defined by the relation (12), the function  $f_u(x)$  representing the part of the graph above the maximum  $x$ , and  $f_l(x)$  representing the part below the maximum  $x$ .

**Exercise 3:** The following are graphs of four more relations for which the method of the article works successfully. The relation for each graph is recorded at the top of each graph. It is left to the interested reader to find an antiderivative  $F(x)$  for each implicitly defined function  $f(x)$  by each relation, the same way the above 6 Examples have been done.



**A new Technique for integration** We end the article by presenting four applications. The first three application below, as well as Remark 1 following the earlier Example 2 show that the Theorem of this article provides a new technique for integration (most likely not explored

before) for specific functions of  $x$ . The first application shows how an easy known integral can be done in a nonconventional way. However, the next two applications are pretty challenging (and likely unknown).

**Application 1** The method can provide a more clearcut way to do the integration

$$\int \sqrt{a^2 - x^2}, 0 \leq |x| < a.$$

The reader may recall that this integration is usually accomplished by the method of substitution of  $x = a \cos \theta$  or

$x = a \sin \theta$ . However, if we apply the relation Theorem of the article to the cartesian relation  $y = \sqrt{a^2 - x^2}$ , representing the upper half circle of radius  $a$ , the polar version of the relation is simply  $\rho = a$ , and therefore the Theorem implies that an antiderivative for  $y = f(x)$  is,

$$\begin{aligned} F(x) &= \frac{xy}{2} - \frac{1}{2} \int_0^{\tan^{-1}(\frac{y}{x})} a^2 d\theta = \\ &= \frac{xy}{2} - \frac{1}{2} a^2 \theta = \frac{xy}{2} - \frac{a^2}{2} \tan^{-1}\left(\frac{y}{x}\right) \\ \frac{xy}{2} - \frac{1}{2} a^2 \theta &= \frac{xy}{2} - \frac{a^2}{2} \tan^{-1}\left(\frac{\sqrt{a^2 - x^2}}{x}\right) \\ &= \frac{x\sqrt{a^2 - x^2}}{2} - \frac{1}{2} \sin^{-1}\left(\frac{x}{a}\right) \end{aligned}$$

Therefore, the general antiderivative for  $y = \sqrt{a^2 - x^2}$  would be

$$F(x) = \frac{x\sqrt{a^2 - x^2}}{2} - \frac{1}{2} \sin^{-1}\left(\frac{x}{a}\right) + C.$$

We end the article by presenting two applications of the method under discussion. The first one, like what mentioned in the Remark on page 8 shows that the method can be used as a new technique for integration, and the second one is to use it to find a trigonometric approximation for the function  $f(x) = \ln x$ .

**Application 2** We now use the to integrate a challenging one, that is the

$$\text{function } y = \sqrt{1 - x^2} + \sqrt{1 + x^2}, |x| < 1.$$

To this end, we first substitute  $x = \rho \cos \theta$  and  $y = \rho \sin \theta$  to convert the above relation between  $x$  and  $y$  into polar form. Squaring both sides after substitution we get  $\rho^2 \sin^2 \theta = 1 - \rho^2 \cos^2 \theta + \sqrt{1 + \cos^2 \theta}$ . This relation can be simplified into  $\rho^2 - 1 = \sqrt{1 + \cos^2 \theta}$ . Squaring both sides again, and then dividing both sides  $\rho^2$ , we conclude  $\rho^2(\theta) = 2 + \cos^2 \theta$ . Next,  $x = 0$  and  $y = \sqrt{2}$  satisfy relation (1) the Theorem implies the following antiderivative for  $y = f(x)$ ,

$$\begin{aligned}
F(x) &= \frac{xy}{2} - \frac{1}{2} \int_0^{\tan^{-1}(\frac{y}{x})} (2 + \cos^2 \theta) d\theta = \\
&= \frac{xy}{2} - \frac{1}{2} \left[ 2\theta + \int_0^{\tan^{-1}(\frac{y}{x})} \cos^2 \theta d\theta \right] = \\
&= \frac{xy}{2} - \frac{1}{2} \left[ \frac{5}{2}\theta + \int_0^{\tan^{-1}(\frac{y}{x})} \frac{\cos 2\theta}{2} d\theta \right] = \\
&= \frac{xy}{2} - \frac{1}{4} \left[ 5 \tan^{-1}\left(\frac{y}{x}\right) + \frac{xy}{(x^2 + y^2)} \right] + C
\end{aligned}$$

Therefore, the required general antiderivative for  $y = \sqrt{1-x^2} + \sqrt{1+x^2}$  is,

$$F(x) = \frac{x\sqrt{1-x^2} + \sqrt{1+x^2}}{2} - \frac{1}{4} \left[ 5 \tan^{-1}\left(\frac{\sqrt{1-x^2} + \sqrt{1+x^2}}{x}\right) + \frac{x\sqrt{1-x^2} + \sqrt{1+x^2}}{\sqrt{1+x^2}} \right] + C$$

A different application is the following.

**Note that**, exactly the same argument can be use to integrate the (two parameter) family of functions  $y = \sqrt{a^2 - bx^2} + \sqrt{a^2 + b^2x^2}$ ,  $b \neq 0$ ,  $0 < |x| < \left|\frac{a}{b}\right|$ . As a time saving hint the polar version of the equation is  $\rho^2(\theta) = \left(2a + \frac{b}{2}\right) + \frac{b}{2} \cos 2\theta$ , which is easily, but patiently integrable. It is left to the reader to find the integral  $\int y dy$ .

**Application 3** (El Ganzo ). We now use the same method in to integrate the function  $y = x^{\frac{3}{2}} \sqrt{x} + \sqrt{2+x^2}$ .

Again, we first substitute  $x = \rho \cos \theta$  and  $y = \rho \sin \theta$  to convert the above relation between  $x$  and  $y$  into polar form. Squaring both sides after substitution and then dividing both sides by  $\rho^2$  implies

$$\sin^2 \theta = \rho \cos^3 \theta (\rho \cos \theta + \sqrt{2 + \rho^2 \cos^2 \theta}), \text{ or}$$

$$\sin^2 \theta - \rho^2 \cos^4 \theta = \rho \cos^3 \theta \sqrt{1 + \cos^2 \theta}.$$

Squaring both sides, and canceling like terms  $\rho^4 \cos^8 \theta$  from the two sides we get,  $\sin^4 \theta - 2\rho^2 \sin^2 \theta \cos^4 \theta = 2\rho^2 \cos^6 \theta$ .

If we now substitute  $1 - \cos^2 \theta$  for  $\sin^2 \theta$  on the left the like terms  $2\rho^2 \cos^6 \theta$  from the two sides cancel, and the result can be expressed as  $\rho^2 = \frac{1}{2} \tan^4 \theta$ .

Since (0,0) is a point on the graph of  $y = f(x)$ , we now apply the Theorem to write the following antiderivative for  $y = x^{\frac{3}{2}} \sqrt{x + \sqrt{2 + x^2}}$ ,

$$F(x) = \frac{xy}{2} - \frac{1}{2} \int_0^{\tan^{-1}(\frac{y}{x})} \tan^4 \theta \, d\theta.$$

To integrate  $\int \tan^4 \theta \, d\theta$  we only need to substitution  $u = \tan \theta$ ;  
 $du = (1 + \tan^2 \theta) \, d\theta$  to get

$$\int \tan^4 \theta \, d\theta = \int \frac{u^4}{1+u^2} \tan^4 \theta \, d\theta = \int (u^2 - 1 + \frac{1}{1+u^2}) \, d\theta = \frac{1}{3} u^3 - u + \tan^{-1} u = \frac{1}{3} \tan^3 \theta - \tan \theta + \tan^{-1}(\tan \theta) = \frac{1}{3} \tan^3 \theta - \tan \theta + \theta.$$

Therefore, the integral of  $y = x^{\frac{3}{2}} \sqrt{x + \sqrt{2 + x^2}}$  will be

$$\begin{aligned} F(x) &= \frac{xy}{2} - \frac{1}{2} \int_0^{\tan^{-1}(\frac{y}{x})} \rho^2(\theta) \, d\theta = \frac{xy}{2} - \frac{1}{2} \int_0^{\tan^{-1}(\frac{y}{x})} \rho^2 = \frac{1}{2} \tan^4 \theta \, d\theta = \\ &= \frac{xy}{2} - \frac{1}{4} \left[ \frac{y^3}{3x^3} - \frac{y}{x} + \tan^{-1}\left(\frac{y}{x}\right) \right] + C. \text{ Therefore,} \\ \int x^{\frac{3}{2}} \sqrt{x + \sqrt{2 + x^2}} \, dx &= \frac{xy}{2} + \frac{y}{4x} \left[ \frac{y^3}{3x^3} - \frac{1}{4} \tan^{-1}\left(\frac{y}{x}\right) \right] + C, \end{aligned}$$

Where substituting  $y = x^{\frac{3}{2}} \sqrt{x + \sqrt{2 + x^2}}$  will present the answer exclusively in terms of  $x$ .

The following is another application of different nature.

**Application 4** Here, we show how the Theorem can be used to find a trigonometric approximation can be obtained for the function  $y = \ln(x)$ .

We first conclude the following corollary from the Theorem:

**Corollary:** For any positive number  $x$ , we have

$$\ln x = - \int_{\frac{\pi}{4}}^{\tan^{-1}(\frac{1}{x^2})} \csc(2\theta) \, d\theta. \quad (*)$$

**Proof:** Consider the function  $f(x) = \frac{1}{x}$  and let  $a = 1$ . Then upon substitutions  $x = \rho \cos \theta$  and  $y = \rho \sin \theta$  in the equation  $y = \frac{1}{x}$  we get  $\rho \sin \theta = \frac{1}{\rho \cos \theta}$ , therefore  $\rho^2 = \frac{1}{\sin \theta \cos \theta} = 2 \csc(2\theta)$ .

Now, setting  $a = 1$  and considering that for any positive number  $x$  we have  $xf(x) = 1$  the formula (1) of the Theorem implies

$$\begin{aligned} F(x) &= \frac{xf(x)}{2} - \frac{1}{2} \int_{\tan^{-1}(\frac{f(1)}{1})}^{\tan^{-1}(\frac{f(x)}{x})} \rho(\theta)^2 \, d\theta = \\ &= \frac{1}{2} - \frac{1}{2} \int_{\frac{\pi}{4}}^{\tan^{-1}(\frac{f(x)}{x})} 2 \csc(2\theta) \, d\theta = \end{aligned}$$

$$\frac{1}{2} - \int_{\frac{\pi}{4}}^{\tan^{-1}(\frac{1}{x^2})} \csc(2\theta) d\theta$$

To show that above proposed antiderivative is indeed an antiderivative of  $y = \frac{1}{x}$  we will show that the condition

$$\frac{d}{dx} \left[ \int_{\tan^{-1}(\frac{f(a)})}{\tan^{-1}(\frac{f(x)})} \rho(\theta)^2 d\theta \right] = xy' - y.$$

of the Theorem holds. To this end considering that  $y' - \frac{1}{x^2}$ , the right side above is simplified as  $x(-\frac{1}{x^2}) - \frac{1}{x} = -\frac{2}{x}$ , and the left side amounts to

$$\begin{aligned} \frac{d}{dx} \left[ \int_{\frac{\pi}{4}}^{\tan^{-1}(\frac{1}{x^2})} 2 \csc(2\theta) d\theta \right] &= 2 \frac{d}{dx} \left[ \frac{1}{2} \ln(\tan(\theta)) \right] \Bigg|_{\frac{\pi}{4}}^{\tan^{-1}(\frac{1}{x^2})} = \\ \frac{d}{dx} \left[ \ln \frac{1}{x^2} \right] &= -2 \frac{d}{dx} \ln x = -\frac{2}{x}. \end{aligned}$$

Hence, the condition (2) of the Theorem holds, and indeed

$$F(x) = \frac{1}{2} - \int_{\frac{\pi}{4}}^{\tan^{-1}(\frac{1}{x^2})} \csc(2\theta) d\theta$$

Is an antiderivative of  $f(x) = \frac{1}{x}$ . Next, since both functions  $\ln x$  and  $F(x) - \frac{1}{2} =$

$-\int_{\frac{\pi}{4}}^{\tan^{-1}(\frac{1}{x^2})} \csc(2\theta) d\theta$  are particular antiderivatives of

the  $f(x) = \frac{1}{x}$  with initial condition  $y(1) = 0$ , they should be equal functions, that is,

$\ln x = F(x) - \frac{1}{2} = -\int_{\frac{\pi}{4}}^{\tan^{-1}(\frac{1}{x^2})} \csc(2\theta) d\theta$ , and the proof of the Corollary is complete.

### **Trigonometric Approximation of $\ln(x)$**

Now for any  $x > 0$ , to obtain the claimed approximation for  $\ln x$  it is only enough to express the definite integral (\*) in the Corollary as the limit of one of the known Riemann sums over the interval  $[\frac{\pi}{4} - \theta_x]$ , say the limit of the right Riemann sum], to get

$$\ln(x) = \left(\frac{\pi}{4} - \theta_x\right) \lim_{n \rightarrow \infty} \sum_{i=1}^n \csc 2\left[\frac{\pi}{4} + \frac{i}{n}(\theta_x - \frac{\pi}{4})\right].$$

Using the identity  $\csc\left(\frac{\pi}{2} - \alpha\right) = \sec \alpha$  the above approximation will look easier, as our claimed approximationas:

$$\ln(x) = \left(\frac{\pi}{4} - \theta_x\right) \lim_{n \rightarrow \infty} \sum_{i=1}^n \sec\left[\frac{2i}{n}\left(\frac{\pi}{4} - \theta_x\right)\right]. \quad (**)$$

We will also get two more similar approximation for  $\ln(x)$ , by expressing the same definite integral as the limits of left and mid-point Riemann sums.