Analytical Geometric Characterizations of Parabolas

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The main objective of this article (The Theorem) is to present two closely related analytical geometric characterizations of parabolas, as represented by quadratic functions. The first characterization simply asserts that for any two points $A\ (a, f(a))$ and $B\ (b, f(b))$ on the graph of a parabola, the slope of the secant segment AB is always the average between the slopes of the tangent lines at A and B to the curves. The second characterization is that the two such described tangent lines always meet (horizontally) right midway between A and B. These two closely related properties, when written in the form of algebraic relations between a and b, turns out to generalize to an interesting assertion (The Proposition) regarding how slopes of two intersecting tangent lines to the graph of a general differentiable function are related to the abscissa of the point of their intersection.

THEOREM: Let $f(x)$ be a ***nonlinear*** differentiable function over the real number line. Then the following conditions are equivalent:

(I) $f(x)$ is a quadratic function; that is $f(x)$ represents a parabola.

(II) For any two points $A\ (a, f(a))$ and $B\ (b, f(b))$ on the graph of $f(x)$, the slope of the secant segment AB is the average between the slopes of the two tangent lines to the graph of $f(x)$ at A and B. In algebraic form this means for any two points $A\ (a, f(a))$ and $B\ (b, f(b))$ on graph of $f(x)$ the following relation holds:

$$[f'(b) + f'(a)]/2 = [f(b) - f(a)]/(b - a). \tag{1}$$

(III) For any distinct points $A\ (a, f(a))$ and $B\ (b, f(b))$ on the graph of $f(x)$, the x-coordinate of the point of intersection of the two tangent lines to the graph of $f(x)$ at points $(a, f(a))$ and $(b, f(b))$ is exactly the average $(a + b)/2$ of the two x-coordinates of the points $(a, f(a))$ and $(b, f(b))$.

PROOF: (I) ⇨(II) Verification of this part is only a routine algebraic manipulation. Let the points $A\ (a, f(a))$ and $B\ (b, f(b))$ be on the graph of a quadratic function $f(x) = c(x - h)^2 + k$. Then $f'(x) = 2c(x - h)$ and we have,

$$f'(a) = 2c(a - h)$$
$$f'(b) = 2c(b - h).$$

Therefore

$$[f'(b)+f'(a)]/2 = [2c(a-h) + 2c(b-h)]/2 = c(a+b-2h)$$

$$\begin{aligned}[f(b)-f(a)]/(b-a) &= [c(b-h)^2 + k - c(a-h)^2 - k]/(b-a)\\ &= c[(b-h)^2 - (a-h)^2/(b-a)\\ &= c[b^2 - a^2) - 2h(b-a)]/(b-a)\\ &= c(b+a-2h)\end{aligned}$$

which shows the two sides of (1) are identical, and thus (II) follows.

(II) ⇨ (III) Let $f(x)$ be a nonlinear differentiable function satisfying the assertion (II). Then, given any two points $A\ (a, f(a))$ and $B\ (b, f(b))$ on the graph of $f(x)$, the respective equations of the tangent lines at these two points are as follows,

$$\begin{aligned}y &= f'(a)(x-a) + f(a)\\ y &= f'(b)(x-b) + f(b)\end{aligned}$$

In order to find the x-coordinate of the point of intersection of the two tangent lines we need to solve the above system of linear equations for x. A simple calculation shows that

$$x = \frac{bf'(b) - af'(a) - [f(b) - f(a)]}{f'(b) - f'(a)}. \tag{2}$$

Since by assumption of part (II), (1) holds, and since we can first rewrite (1) in the form

$$[f(b) - f(a)] = (b-a)\frac{[f'(a) + f'(b)]}{2},$$

upon substitution of the above right hand side for $[f(b) - f(a)]$ in the numerator of the fraction in (2) we get

$$x = \frac{bf'(b) - af'(a) - (b-a)[f'(a) + f'(b)]/2}{f'(b) - f'(a)}$$

$$= \frac{2bf'(b) - 2af'(a) - (b-a)[f'(a) + f'(b)]}{2[f'(b) - f'(a)]} = \frac{bf'(b) - af'(a) - bf'(a) + af'(b)]}{2[f'(b) - f'(a)]}$$

$$= \frac{[f'(b) - f'(a)](a+b)}{2[f'(b) - f'(a)]} = \frac{(a+b)}{2},$$

which means (III) follows.

(III) ⇨ (I) To this end, assume $f(x)$ is a differentiable function satisfying assertion (III).

Then, again, for two arbitrary points $A\ (a, f(a))$ and $B\ (b, f(b))$ on the graph of $f(x)$, equations of the tangent lines at these two points are,

$$y = f'(a)(x-a) + f(a)$$
$$y = f'(b)(x-b) + f(b).$$

Since by assumption of part (III) the x-coordinate of the point of intersection of the above two lines is $x = (a+b)/2$, when substituting this abscissa in the above two equations, the two right hand sides should be identical. That is,

$$f'(a)(\frac{a+b}{2} - a) + f(a) = f'(b)(\frac{a+b}{2} - b) + f(b),$$

or

$$f'(a)(\frac{b-a}{2}) + f(a) = f'(b)(\frac{a-b}{2}) + f(b).$$

Or

$$f'(a)(b-a) + 2f(a) = f'(b)(a-b) + 2f(b). \tag{3}$$

Since (3) should hold for any two real numbers a and b, we can keep the parameter a fixed, say $a = 0$ and let $b = x$ vary over all ***nonzero*** real numbers. Then (3) becomes

$$xf'(0) + 2f(0) = - xf'(x) + 2f(x).$$

Using the usual notations $y = f(x)$ and $y' = f(x)$ we obtain a linear differential equation as follows

$$xy' - 2y = - xf'(0) - 2f(0).$$

We now divide both sides of this last equation by x^3, and write the equation as follows

$$\frac{x^2y' - 2xy}{x^4} = \frac{-f'(0)}{x^2} - \frac{2f(0)}{x^3}.$$

Or

$$\frac{d}{dx}(\frac{y}{x^2}) = \frac{-f'(0)}{x^2} - \frac{2f(0)}{x^3},$$

which (upon integration of both sides) implies

$$\frac{y}{x^2} = \frac{f'(0)}{x} + \frac{f(0)}{x^2} + C.$$

Multiplying both sides by x^2 implies that $y = f(x)$ must be the quadratic function

$$f(x) = f'(0)x - f(0) + Cx^2$$

This completes the proof of the theorem.

Note that in the above proof C can not be zero, otherwise the function will be linear, contrary to the assumption of the theorem.

REMARK Equivalence of parts (II) and (III) in the above theorem can be generalized into the following interesting skew-type property between two tangent lines to the graph of any function, ***when they intersect***. Here I take the point of view that the number $(a + b)/2$ in part (III) and the number $[f'(a) + f'(b)]/2$ in part (II) are only one of the zillions ***corresponding*** linear combinations of the respective numbers a and b in part (III) and of numbers $f'(a)$ and $f'(b)$ in part (II).

PROPOSITION Let $f(x)$ be a differentiable function and $A(a, f(a))$, $B(b, f(b))$ be arbitrary points on its graph, with their respective intersecting tangent lines T_a and T_b. If c denotes the abscissa of the point of intersection of T_a and T_b, then for any given real number s, the relation $c = sa + (1 - s)b$ holds if and only if

$$\frac{f(b) - f(a)}{b - a} = (1 - s)f'(a) + sf'(b).$$

This means the same linear combination expressing the slope of the secant segment AB in terms of the respective slopes $f'(a)$ and $f'(b)$ of T_a and T_b will determine how the abscissa of the point of intersection of the lines T_a and T_b can be expressed in terms of a and b (in a skew manner as seen above).

PROOF: Since (as we saw in the proof (II)⇨(III) of the Theorem) the abscissa of the point of intersection of T_a and T_b is

$$c = \frac{bf'(b) - af'(a) - f(b) + f(a)}{f'(b) - f'(a)},$$

the relation

$$\frac{bf'(b) - af'(a) - f(b) + f(a)}{f'(b) - f'(a)} = sa + (1 - s)b$$

can be cross-multiplied and rearranged to be converted into

$$\frac{f(b) - f(a)}{b - a} = (1 - s)f'(a) + sf'(b).$$

Since all the steps of the above proof are reversible, the assertion "if and only if" of the proposition follows.

The above proposition provides an indirect way of finding the point of intersection of two intersecting tangent lines to a differentiable function, once you know the slopes of the tangent lines at the two points, as seen in the following example.

EXAMPLE: Let $f(x) = e^x$, and consider the two points $A\,(a, f(a)) = (1, e)$, $B(b, f(b)) = (2, e^2)$ on the graph of this function. Here, $f'(1) = e$ and $f'(2) = e^2$. Noting that the graph of $f(x)$ obviously suggests that T_a and T_b intersect, and since

$$\frac{f(b) - f(a)}{b - a} = \frac{e^2 - e}{2 - 1} = e^2 - e = s \times f'(1) + (1 - s) \times f'(2) = s \times e + (1 - s) \times e^2$$

with $s = \frac{1}{e-1}$, the proposition implies the abscissa of the point of intersection of T_a and T_b should be the ***skewed-corresponding*** combination $c = (1-s)\times e + s\times e^2 = \frac{e}{e-1}$.

Therefore $(e/(e-1), e^2/(e-1))$ would be the point of intersection of T_a and T_b. This fact can also be confirmed by directly solving the following system of linear equations.

$$\begin{cases} T_a\colon y = e(x-1)+e \\ T_b\colon y = e^2(x-2)+e^2 \end{cases}.$$