Trigonometric Approximation of Logarithmic Function Ln(x)

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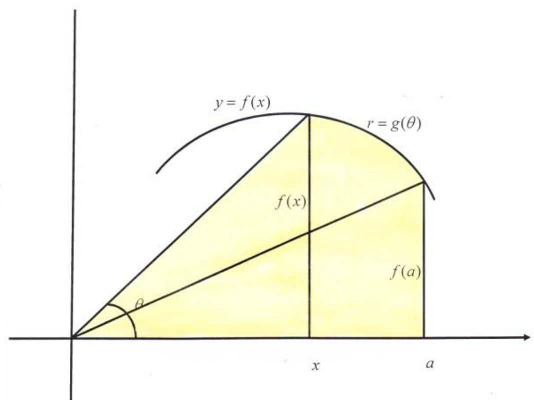
I start by recalling the first Theorem in item #1 of this same Calculus 2 section of the website on the definite integral version of the method of Implicit Integration,

Theorem Let $y = f(x)$ be a differentiable function defined over the open interval $(0,\infty)$ and let $r = g(\theta)$ be the polar representation of the curve of $y = f(x)$ obtained upon substitutions $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Then, for any $a > 0$ the following will be an anti-derivative for $f(x)$,

$$F(x) = \frac{xf(x)}{2} - \frac{1}{2}\int_{tan^{-1}[\frac{f(a)}{a}]}^{tan^{-1}[\frac{f(x)}{x}]} r(\theta)^2 d\theta.$$



Now we apply the above Theorem to conclude the following corollary, representing the function $Ln\,(x)$ as follows,

Corollary: $$Ln\,(x) = -\int_{\pi/4}^{\tan^{-1}(1/x^2)} \csc(2\theta)\,d\theta. \qquad (1)$$

Proof: Set $f(x) = 1/x$, and $a = 1$. Applying substitutions $x = r\cos(\theta)$ and $y = r\sin(\theta)$ in the equation $f(x) = 1/x$, it follows that $r\sin(\theta) = \dfrac{1}{r\cos(\theta)}$, and therefore

$$r^2 = g(\theta)^2 = 1/\sin(\theta)\cos(\theta) = 2\csc(2\theta).$$

Also considering that in this particular case for every $x>0$ we have $x\,f(x)=1$, as in above proposition, the following $F_1(x)$ is the anti-derivative for $f(x)=1/x$ satisfying $F_1(1)=0$,

$$F_1(x)=\frac{x\,f(x)}{2}-\frac{a\,f(a)}{2}-\frac{1}{2}\int_{\pi/4}^{\tan^{-1}[f(x)/x]} 2\csc(2\theta)\,d\theta=-\int_{\pi/4}^{\tan^{-1}[f(x)/x]} 2\csc(2\theta)\,d\theta .$$

Since $Ln(x)$ is also an anti-derivative for $f(x)=1/x$ satisfying $Ln(1)=0$, the result follows.

<u>Note that</u>, the above integral representation (1) for $Ln(x)$ in the Corollary can also be concluded from the Fundamental Theorem of Calculus, in conjunction with the chain rule for differentiation. That is, simply by showing that for any $x>0$ the derivative of right side of (1) is $1/x$.

Riemann sums for the definite integral on the right hand side of (1) in above Corollary can be used to globally approximate logarithmic function $Ln\,(x)$.

For any given x, assuming $\tan^{-1}(1/x^2)=\theta$, and expressing the right hand side definite integral in above (1) in the form of limit of the right Riemann sum, we obtain the following global trigonometric approximation for $Ln(x)$,

$$Ln(x)=(\frac{\pi}{4}-\theta)\ \lim_{n\to\infty}\sum_{i=1}^{n}\csc[2\theta+\frac{2i}{n}(\frac{\pi}{4}-\theta)]/n \quad . \qquad (2)$$

For those values of $Ln\,(x)$ for which $\theta=\tan^{-1}(1/x^2)$'s are among particular fractions of π the above expansion (2) will provide handsome trigonometric approximations converging to $Ln\,(x)$. The following are only three examples of such approximations, for three specific values of x.

<u>Example 1:</u> For $x=\sqrt[4]{3}$, we have $\theta=\tan^{-1}(1/x^2)=\frac{\pi}{6}$, and (2) implies

$$Ln(3)=\frac{\pi}{3}\lim_{n\to\infty}[\csc(\frac{\pi}{3}+\frac{\pi}{6n})+\csc(\frac{\pi}{3}+\frac{2\pi}{6n})+\ldots+\csc(\frac{\pi}{3}+\frac{n\pi}{6n})]/n$$

Note that, say for $n=4$ the above sequence provides approximations 1.0803 and for $Ln(3)$, which is an approximation for $Ln(3)$ correct to 4 decimals.

Using the trigonometric identity $\cos(\frac{\pi}{2}-\alpha)=\sin(\alpha)$, the above approximation will look prettier in the eye, when expressed as the following limit

$$\lim_{n\to\infty}[\sec\frac{(n-1)\pi}{6n}+\sec\frac{(n-2)\pi}{6n}+\ldots+\sec(0)]/n\ =\ \frac{3}{\pi}Ln(3) \quad .$$

Example 2: For $x = 1/\sqrt{\sqrt{2}-1}$ we have $\theta = \tan^{-1}(1/x^2) = \dfrac{\pi}{8}$ and (2) implies

$$Ln(\sqrt{2}-1) = -\frac{\pi}{4}\lim_{n\to\infty}[\csc(\frac{\pi}{4}+\frac{\pi}{4n}) + \csc(\frac{\pi}{4}+\frac{2\pi}{4n}) + \ldots + \csc(\frac{\pi}{4}+\frac{n\pi}{4n})]/n,$$

which again using the identity $\cos(\dfrac{\pi}{2}-\alpha) = \sin(\alpha)$ can be expressed better as ,

$$\lim_{n\to\infty}[\sec\frac{(n-1)\pi}{4n} + \sec\frac{(n-2)\pi}{4n} + \ldots + \sec(0)]/n = -\frac{4}{\pi}Ln(\sqrt{2}-1).$$

Example 3: For $x = 1/\sqrt{2-\sqrt{3}}$ we have $\theta = \tan^{-1}(1/x^2) = \dfrac{\pi}{12}$, and (2) implies,

$$Ln(2-\sqrt{3}) = -\frac{\pi}{3}\lim_{n\to\infty}[\csc(\frac{\pi}{6}+\frac{\pi}{3n}) + \csc(\frac{\pi}{6}+\frac{2\pi}{3n}) + \ldots + \csc(\frac{\pi}{6}+\frac{n\pi}{3n})]/n,$$

which can also be expressed as

$$\lim_{n\to\infty}[\sec\frac{(n-1)\pi}{3n} + \sec\frac{(n-2)\pi}{3n} + \ldots + \sec(0)]/n = -\frac{3}{\pi}Ln(2-\sqrt{3}).$$