Remarkable Combinatorics Identities

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The n! Identity:

Sometime around November/December 2012 while I was trying to come up with a direct formulation of the nth derivative of a function (see the article with the same title in Calculus part of my website) I concluded a remarkable combinatorial identity for $n!$ in terms of the familiar combinations notation $\binom{n}{j}$ (i.e., combinations of n objects taken j at a time). Here is the identity, followed by a easy example to justify it momentarily,

$$n! = n^n\binom{n}{0} - (n-1)^n\binom{n}{1} + (n-2)^n\binom{n}{2} + \ \ldots\ +(-1)^{(n-1)}.1^n\binom{n}{n-1}$$

Or, in sigma notation,

$$n! = \sum_{j=o}^{n-1}(-1)^j(n-j)^n\binom{n}{j}$$

Example 1 When n = 4 a good exercise for a BC Pre-Calculus 12 would be to show,

$$4! = 4^4\binom{4}{0} - 3^4\binom{4}{1} + 2^4\binom{4}{2} - 1^4\binom{4}{3} = 4!$$

A proof of the identity can be presented by induction on n, as did an APCalculus student of mine as a challenging project around June 2013.

I needed the way I came up with this identity was when I was trying to conclude a formulation for n^{th} derivative of a function by using L'Hopita's rule n times (see the article title " a direct formula … " in Calculus section of this website.

The 2^n Identity:

The following identity expresses 2^n in terms of powers of its reciprocal $\frac{1}{2}$.
For any positive integer n, we have

$$2^n = \binom{n}{n}\left(\frac{1}{2^0}\right) + \binom{n+1}{n}\left(\frac{1}{2^1}\right) + \binom{n+2}{n}\left(\frac{1}{2^2}\right) + \binom{n+3}{n}\left(\frac{1}{2^3}\right) + \ \ldots\ + \binom{2n}{n}\left(\frac{1}{2^n}\right)$$

Or, in sigma notation

$$2^n = \sum_{j=o}^{n}\binom{n+j}{n}\frac{1}{2^j}$$

I must have proved this by induction ages ago.

Example 2 When n = 4 another good exercise for students would be to show,

$$2^4 = \binom{4}{4}\left(\frac{1}{2^0}\right) + \binom{5}{4}\left(\frac{1}{2^1}\right) + \binom{6}{4}\left(\frac{1}{2^2}\right) + \binom{7}{4}\left(\frac{1}{2^3}\right) + \binom{8}{4}\left(\frac{1}{2^4}\right)$$

<u>The $(\boldsymbol{n}+\mathbf{1})^{\boldsymbol{n}}$ Identity</u>:

The following identity expresses $(\boldsymbol{n}+\mathbf{1})^{\boldsymbol{n}}$ in terms $\boldsymbol{n}^{\boldsymbol{th}}$ powers of successive lower terms. For any positive integer n, we have

$$(n+1)^n = n^n\binom{n+1}{1} - (n-1)^n\binom{n+1}{2} + (n-2)^n\binom{n+1}{3} + \ \dots + \ (-1)^{n-1}.1^n.\binom{n+1}{n}$$

Or, in sigma notation,

$$(n+1)^n = \sum_{j=0}^{n-1}(-1)^j(n-j)^n\binom{n+1}{j+1}$$

<u>Example 3</u> When n =4 it can be verified that,

$$5^4 = 4^4\binom{5}{1} - 3^4\binom{5}{2} + 2^4\binom{5}{3} - 1^4\binom{5}{4}.$$