# A Method of Implicit Integration

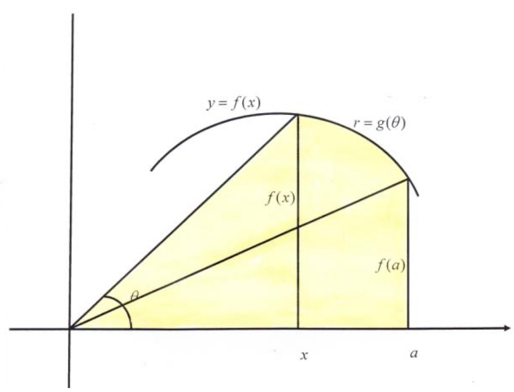
***Ali Astaneh,Ph.D(Lon), Vancouver, BC.***

**Abstract**

In this article we Introduce a method that enables us to find antiderivatives for functions implicitly defined through a certain class of relations $R(x, y) = 0$ on the $xy$ - coordinate system. As far my past decades of teaching Calculus at different levels are concerned, most likely the method is the closest one can get to integrate an implicitly defined a function.
I came up with the metod early 2012 while teaching AP-Calculus at Prince of Wales Secondary, and afterwards I presented it at the October 2012 BC Association of Math Teachers' conference. The method has not been mentioned in any Calculus textbook, so it is reasonable to guess that it is original! Similar to the case of implicit differentiation here too the anti-derivatives are obtained in terms of both variables $x$ and $y$, in general.

**Theorem** [A. Astaneh] Let $y = f(x)$ be a differentiable function defined over the open interval $(0,\infty)$ and let $r = g(\theta)$ be the polar representation of a curve $y = f(x)$ upon substitutions $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Note that here, if the function $y = f(x)$ is defined by an implicit relation, say $R(x, y) = 0$, it is then understood that $r = g(\theta)$ is the polar version of this implicit relation. Then, for any $a > 0$ the following will give an anti-derivative for $f(x)$, which in general (and eventually) will be expressed in terms of both $x$ and $y$.

$$F(x) = \frac{xf(x)}{2} - \frac{1}{2}\int_{tan^{-1}[\frac{f(a)}{a}]}^{tan^{-1}[\frac{f(x)}{x}]} r(\theta)^2 d\theta.$$




**Proof:** In the above diagram, let $F_1(x)$ be the anti-derivative for $y = f(x)$ defining the area under the graph of $f(x)$ over the interval $[x, a]$, and let $A_x$ denote this area. Also, let $A_\theta$ denote the polar area enclosed by the origin, the polar curve $r = g(\theta)$, and the polar rays of line defined by equations

$\theta_1 = \tan^{-1}(f(a)/a)$, $\theta_2 = \tan^{-1}(f(x)/x)$. Then the shaded area in the above diagram can be represented by the flowing equal expressions,

$$\frac{xf(x)}{2} + A_x = \frac{a\,f(a)}{2} + A_\theta . \qquad (*)$$

Using the anti-derivative $F_1(x)$, for $f(x)$, and applying the Fundamental Theorem of Calculus upon it to find the area under the curve of $y = f(x)$ over the interval $[x, a]$, and also using the popular formula

$$\frac{1}{2}\int_{\theta_1}^{\theta_2} r(\theta)^2\, d\theta$$

to express the , the above equation (*) can be expressed as

$$\frac{x\,f(x)}{2} + F_1(a) - F_1(x) = \frac{a\,f(a)}{2} + \frac{1}{2}\int_{\tan^{-1}[f(a)/a]}^{\tan^{-1}[f(x)/x]} r(\theta)^2\, d\theta .$$

Since by assumption $F_1(a) = 0$, it follows,

$$F_1(x) = \frac{x\,f(x)}{2} - \frac{a\,f(a)}{2} - \frac{1}{2}\int_{\tan^{-1}[f(a)/a]}^{\tan^{-1}[f(x)/x]} r(\theta)^2\, d\theta ,$$

Therefore

$$F(x) = F_1(x) + \frac{af(a)}{2} = \frac{xf(x)}{2} - \frac{1}{2}\int_{tan^{-1}[\frac{f(a)}{a}]}^{tan^{-1}[\frac{f(x)}{x}]} r(\theta)^2 d\theta.$$

is an anti-derivative for $y = f(x)$, and the proof is complete.

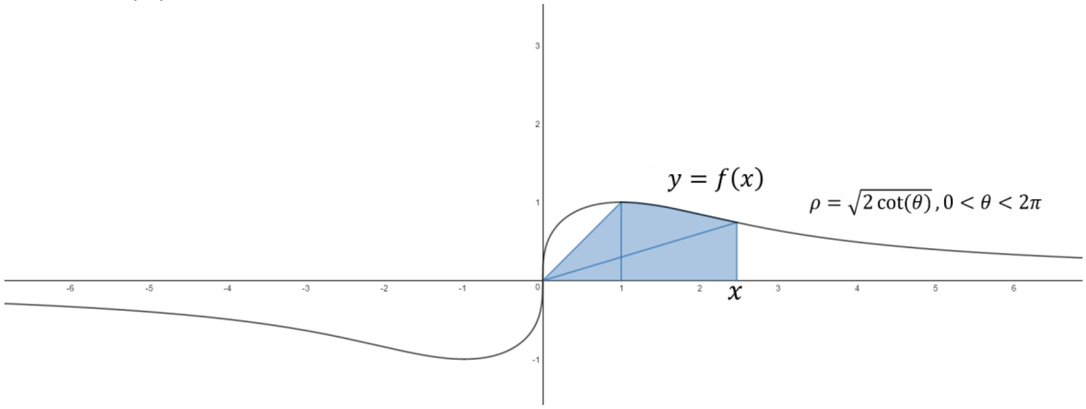
Given a specific implicit relation, and when the real number $a$ is pre-selected, than the above Theorem works like it does in the following four examples of different nature.

**<u>Example 1:</u>** Let $y = f(x)$ be the function implicitly defined by the relation

$$y(x^2 + y^2) = 2x \ , \ \ x \ge 0, y \ge 0 . \qquad (1)$$

Find an anti-derivative $F(x)$ for the function $y = f(x)$ with initial condition $F(1) = 0$.

**<u>Analytic Solution:</u>** Upon polar substitutions of $x = \rho\cos(\theta)$ and $y = \rho\sin(\theta)$ in (1), the relation is converted to the polar form $\rho^2 = 2\cot(\theta)$, or $\rho = \sqrt{2\cot(\theta)}$ withthe following polar graph, which at the same time represents graph of relation (1),




Now let $F(x)$ be the anti-derivative for $y = f(x)$ defining the area under its graph over the interval $[1, x]$, and let $A_x$ denote that area. Also, let $A_\theta$ denote the polar area between the polar curve, the origin, and between arrays of lines $\theta = \pi/4$ and $\theta = \tan^{-1}[f(x)/x]$ . Then the above figure shows the validity of the following relation between the two areas $A_x$ and $A_\theta$ ,

$$\frac{1f(1)}{2} + A_x = \frac{x\,f(x)}{2} + A_\theta .$$

By the Fundamental Theorem of Calculus, and the popular polar formula for the polar area $A_\theta$, the above relation implies,

$$\frac{1}{2} + F(x) - F(1) = \frac{x\,f(x)}{2} + \frac{1}{2}\int_{\tan^{-1}[f(x)/x]}^{\pi/4} \rho(\theta)^2\,d\theta ,$$

$$F(x) = \frac{x\,f(x)}{2} - \frac{1}{2} + \frac{1}{2}\int_{\tan^{-1}[f(x)/x]}^{\pi/4} 2\cot(\theta)d\theta ,$$

$$F(x) = \frac{x\,f(x)}{2} - \frac{1}{2} + \ln(\sin\theta)\Big|_{\tan^{-1}[f(x)/x]}^{\pi/4} ,$$

$$F(x) = \frac{x\,f(x)}{2} - \frac{1}{2} - \frac{1}{2}\ln(2) - \ln\frac{y}{\sqrt{x^2 + y^2}}$$

$$F(x) = \frac{x\,f(x)}{2} + \ln(\frac{\sqrt{(x^2 + f(x)^2)}}{\sqrt{2}\,f(x)}) - \frac{1}{2}$$

Therefore the solution is $F(x)=\frac{x\,y}{2}+\ln(\frac{\sqrt{(x^2+y^2)}}{\sqrt{2}\,y})-\frac{1}{2}$, where $y=f(x)$ is the function implicitly defined in (1).

**Algebraic Verification of the Solution**: It is enough to show that the derivative of the right hand side of the equation $F(x)=\frac{x\,y}{2}+\ln(\frac{\sqrt{(x^2+y^2)}}{\sqrt{2}\,y})-\frac{1}{2}$ is $y=f(x)$.

To this end, in the following, when trying to simplify the derivative of the right hand side of the equation we will make use the original implicit relation (1). That is, when the expression $y(x^2+y^2)$ shows up as the denominator of a fraction we will replace it by$2x$.

$$\frac{d}{dx}\left[\frac{xy}{2}+ln\left(\frac{\sqrt{x^2+y^2}}{\sqrt{2}\;y}\right)-\frac{1}{2}\right]=\frac{d}{dx}\left[\frac{xy}{2}+\frac{1}{2}\,ln(x^2+y^2)-ln\,y\,-\ln 2-\frac{1}{2}\right]$$

$$=\frac{y+xy'}{2}+\frac{2x+2yy'}{2(x^2+y^2)}-\frac{y'}{y}$$

$$=\frac{y+xy'}{2}+\frac{y\left(x+yy'\right)-y'(x^2+y^2)}{y(x^2+y^2)}$$

$$=\frac{y+xy'}{2}+\frac{y\left(x+yy'\right)-y'(x^2+y^2)}{2x}$$

$$\frac{xy+x^2y+xy+y^2y'-x^2y-y^2y'}{2x}=y$$

**Example 2:** Let $y=f(x)$ be the function implicitly defined by the relation

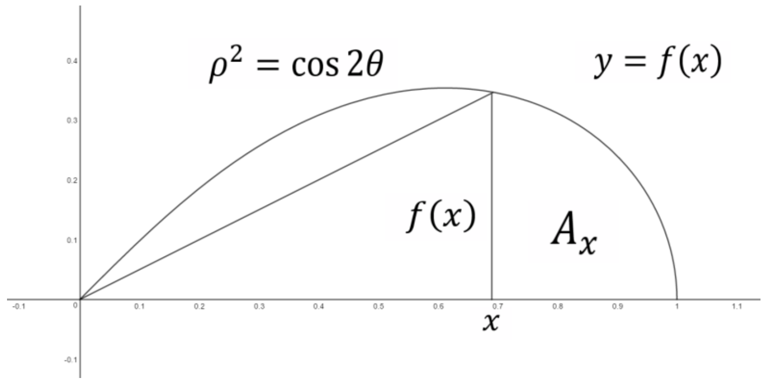
$$(x^2+y^2)^2=x^2-y^2\;,\;\;0\le x\le 1,\;y\ge 0. \quad (2)$$

Find an anti-derivative for $y=f(x)$ satisfying $F(1)=0$.

**Analytical solution:** Substituting $x=\rho\cos\theta$ and $x=\rho\sin\theta$ in (2), the Cartesian relation is converted into a polar relation as follows

$$(x^2+y^2)^2=x^2-y^2\rightarrow\rho^4=\rho^2\cos^2\theta-\rho^2\sin^2\theta=\rho^2\cos2\theta\rightarrow\;\rho^2=\cos2\theta.$$

The conditions $0\le x\le 1$ and $y\ge 0$ mean, here we are concerned with the upper right part of the Lemniscate curve $\rho^2=\cos2\theta$ as seen below,




Let $F(x) = \int_x^1 f(x)dx$, $F(1) = 0$, be the desired anti-derivative for $y = f(x)$ defining the area under the graph over the interval $[x, 1]$, for $x > 0$, and let $A_x$ denote that area under the curve $y = f(x)$ over the interval $[x, 1]$. Also, let

$$A_\theta = \frac{1}{2}\int_0^{\tan^{-1}[f(x)/x]} \rho(\theta)^2 \, d(\theta)$$

be the polar area enclosed by the Lemniscate curve, the $x$-axis, and the lines represented by the polar rays $\theta = 0$ and $\theta = \tan^{-1}[f(x)/x]$. Then, as seen in the diagram, the following relation holds between the two areas $A_x$ and $A_\theta$,

$$A_x = A_\theta - \frac{x\, f(x)}{2}.$$

By the Fundamental Theorem of Calculus the left hand side is just $A_x = F(1) - F(x)$. Therefore, also using the polar definite integral formula for the area $A_\theta$, the above relation can be expressed as

$$F(1) - F(x) = \frac{1}{2}\int_0^{\tan^{-1}[f(x)/x]} \cos(2\theta)\, d\theta \; - \frac{x\, f(x)}{2}.$$

Hence, considering $F(1) = 0$, we have

$$F(x) = \frac{x\, f(x)}{2} - \frac{1}{4}[\sin(2\theta)]\Big|_0^{\tan^{-1}[f(x)/x]}.$$

Or,

$$F(x) = \frac{xf(x)}{2} - \frac{xf(x)}{2[x^2 + f(x)^2]}.$$

Therefore the solution is $F(x) = \dfrac{x\,y}{2} - \dfrac{xy}{2(x^2 + y^2)}$, where $y = f(x)$ is the function implicitly defined in (2).

**Algebraic Verification of the Solution :** Again, it is enough to show that the derivative of the right hand side of $F(x) = \frac{x\,y}{2} - \frac{xy}{2(x^2+y^2)}$ is $y = f(x)$. To this end, this time in our verification when the expression $(x^2 + y^2)^2$ shows up as a denominator of a fraction we will make use the assumed original implicit relation (2) and replace it by $(x^2 - y^2)$ .

$$\frac{d}{dx}\left[\frac{xy}{2} - \frac{xy}{2(x^2+y^2)}\right] = \frac{1}{2}(y + xy') - \frac{1}{2}\frac{(y+xy')(x^2+y^2) - xy(2x+2yy')}{(x^2+y^2)^2} =$$

$$\frac{1}{2}(y+xy') - \frac{1}{2}\frac{xy'(x^2-y^2) - y(x^2-y^2)}{(x^2-y^2)} = \quad .$$

$$\frac{1}{2}(y+xy') - \frac{1}{2}(xy' - y) = y.$$

**Remark:** Incidentally the implicit relation $(x^2+y^2)^2 = x^2 - y^2$ in above Example 2 can be solved for $y \geq 0$, but in a not so handsome expression as follows,

$$y = \frac{1}{\sqrt{2}}\sqrt{\sqrt{8x^2+1} - (2x^2+1)}\,.$$

Therefore, the above solution to Example (2) implies we have been able to explicitly integrate the rather complicated looking integral,

$$\int \frac{1}{\sqrt{2}}\sqrt{\sqrt{8x^2+1} - (2x^2+1)}\; dx, \quad 0 < x \leq 1,$$

as Example 2 implies that the solution is,

$$\int \frac{1}{\sqrt{2}}\sqrt{\sqrt{8x^2+1} - (2x^2+1)}\; dx = \frac{xf(x)}{2} - \frac{xf(x)}{2[x^2 + f(x)^2]} + C,$$

Where $f(x) = \frac{1}{\sqrt{2}}\sqrt{\sqrt{8x^2+1} - (2x^2+1)}$ happens to be the integrand itself. Upon substitution of this expression for $f(x)$, and algebraic simplification, the above integral is obtained as,

$$\int \sqrt{\sqrt{8x^2+1} - (2x^2+1)}\; dx = \frac{x\sqrt{\sqrt{8x^2+1} - (2x^2+1)}\,(\sqrt{8x^2+1} - 3)}{2(\sqrt{8x^2+1} - 1)} + C.$$

The author, for one, doesn't recall any previously known integration technique to handle the above explicit integral. Consider this as a side job bonus from the method of implicit integration presented in this article.

**Example 3** Let $y = f(x)$ be the function implicitly defined by the relation

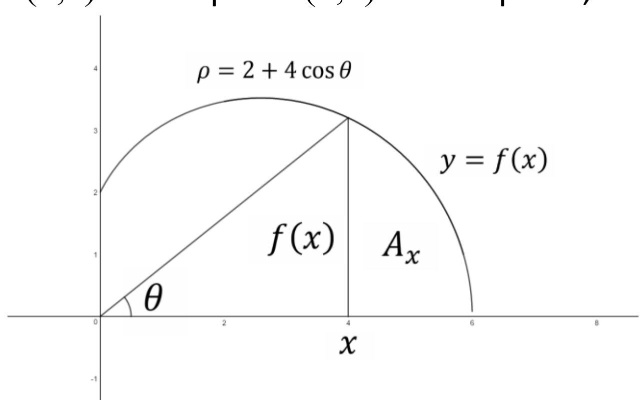
$$(x^2 + y^2 - 4x)^2 = 4(x^2 + y^2)\ ,\ \ 0 < x \leq 6,\ y \geq 0. \quad (3)$$

Find the anti-derivative for $y = f(x)$ satisfying $F(6) = 0$ .

**Analytical solution:** Again, substituting $x = \rho\cos\theta$ and $x = \rho\sin\theta$ in (3), the Cartesian relation is converted into a polar relation as follows,

$$(\rho^2 - 4\rho\cos\theta)^2 = 4\rho^2 \quad \rightarrow \quad (\rho - 4\cos\theta)^2 = 4 \quad \rightarrow \rho = 4\cos\theta \pm 2.$$

Since the case $\rho = 2 - 4\cos\theta$ implies $x = \rho\cos\theta = \cos(\theta)\,(2 - 4\cos(\theta))$ from which the value $x = 6$ will never be obtained for $x$, the requirement $F(6) = 0$ implies that here we are concerned with the polar equation $\rho = 2 + 4\cos\theta$ whose graph is part of the Limacon in the first quadrant curving counter-clockwise from point $(6,0)$ to the point $(0,2)$ on the plane, as seen below




Again, let $F(x)$ be the anti-derivative for $y = f(x)$ defining the area under its graph over the interval $[x, 6]$, and let $A_x$ denote that area. Also, let $A_\theta$ denote the polar area enclosed by the Lemicon, the $x$-axis, and the line with the angle of inclination $\theta = \tan^{-1}[f(x)/x]$. Then the above figure shows the validity of the following relation between the two areas $A_x$ and $A_\theta$ ,

$$A_x = A_\theta - \frac{x\,f(x)}{2}.$$

By the Fundamental Theorem of Calculus, and the polar coordinate formula for $A_\theta$, the above relation can be expressed as

$$F(6) - F(x) = \frac{1}{2}\int_0^{\tan^{-1}[f(x)/x]} [2 + 4\cos(\theta)]^2\, d\theta - \frac{x\,f(x)}{2}.$$

Therefore, considering $F(6)=0$, we have

$$F(x)=\frac{x\,f(x)}{2}-2\int_0^{\tan^{-1}[f(x)/x]}[1+4\cos(\theta)+4\cos^2(\theta)]\,d\theta \quad .$$

Or,

$$F(x)=\frac{x\,f(x)}{2}-2\int_0^{\tan^{-1}[f(x)/x]}[1+4\cos(\theta)+2\cos(2\theta)+2]\,d\theta \ .$$

Hence,

$$F(x)=\frac{x\,f(x)}{2}-2\,[\,3\theta+4\sin(\theta)+\sin(2\theta)]\Big|\Big|_{0}^{\tan^{-1}[f(x)/x]},$$

$$F(x)=\frac{x\,\mathrm{y}}{2}-2\,[\,3\tan^{-1}(\frac{y}{x})+\frac{4y}{\sqrt{x^2+y^2}}+\frac{2xy}{x^2+y^2}].$$

The Algebraic verification of the solution for Example 3 is left to the reader.

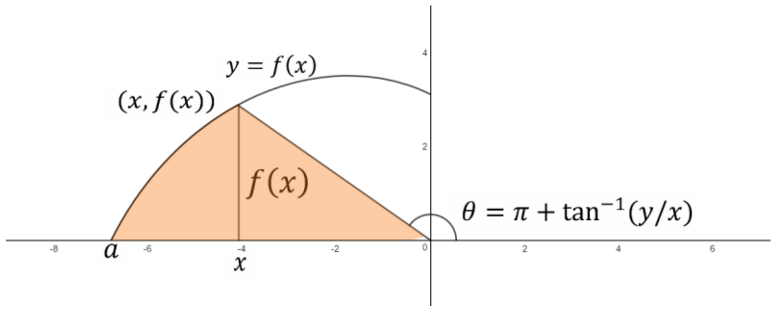
<u>Note that, in the analytical solution for the following example the letter $r$ has been used instead of $\rho$.</u>

**<u>Example 4:</u>** Find an anti-derivative for the function $y=f(x)$ implicitly defined by

$$\ln(\frac{x^2+y^2}{2})=\pi+\tan^{-1}(y/x),\ \ x\in[-\sqrt{2}\,e^{\pi/2},0),[0,\sqrt{2}\,e^{\pi/4}). \qquad (4)$$

<u>Note that</u>, the point with coordinates $(-\sqrt{2}\,e^{\pi/2},0)$ is an $x$-intercept of $y=f(x)$.

**<u>Analytical Solution:</u>** Let $F_1(x)=\int_a^x f(t)dt$ be the area under the graph of $f(x)$ and directly above the interval $[a,x]$, where $a=-\sqrt{2}\,e^{\pi/2}$ is the $x$-intercept of $f(x)$. Then by Fundamental Theorem of Calculus $F_1(x)$ is an anti-derivative for $f(x)$. Moreover, the coloured area shown in the diagram below can be expressed as $F_1(x)-\frac{xf(x)}{2}$. (Note that for $x\in[-6,-1]$ the triangular part of the coloured area is $-xf(x)/2$, because $x<0$).




On the other hand, with the selection of the polar argument $\theta$ as $\theta = \pi + \tan^{-1}(f(x)/x)$, we know from the same coloured polar area is simply $\frac{1}{2}\int_{\pi+\tan^{-1}[f(x)/x]}^{\pi} r(\theta)^2 \, d\theta$, where $r(\theta)$ is the polar representation of the relation (4), in both quadrants I and II, and hence that of the function $y = f(x)$ as well. Therefore we have

$$F_1(x) - \frac{xf(x)}{2} = \frac{1}{2}\int_{\pi+\tan^{-1}[f(x)/x]}^{\pi} r(\theta)^2 \, d\theta. \qquad (*)$$

Next, it is easy to check that upon polar substitutions $x = r\cos(\theta)$ and $y = r\sin(\theta)$, and $x^2 + y^2 = r^2$ in (4) the relation is simplified into $r^2 = 2e^{\theta}$. Substituting this in (*) we get,

$$F_1(x) - \frac{xf(x)}{2} = \frac{1}{2}\int_{\pi+\tan^{-1}[f(x)/x]}^{\pi} 2e^{\theta} \, d\theta = e^{\theta}\Big|_{\pi+\tan^{-1}[f(x)/x]}^{\pi} = e^{\pi} - e^{\pi+\tan^{-1}[f(x)/x]}.$$

Therefore, $F_1(x) = \frac{xf(x)}{2} - e^{\pi+\tan^{-1}[f(x)/x]}$ is an anti-derivative for $y = f(x)$.

<u>**Algebraic verification of the solution:**</u> It is enough to show that the derivative of the right hand side of $F_1(x) = \frac{xf(x)}{2} - e^{\pi+\tan^{-1}[f(x)/x]}$ is just $f(x)$, so here we go

$$\frac{d}{dx}\left[\frac{x\,f(x)}{2} - e^{\pi+\tan^{-1}[f(x)/x]}\right] = \frac{1}{2}(f(x) + x\,f'(x)) - \frac{x\,f'(x) - f(x)}{x^2 + f(x)^2} e^{\pi+\tan^{-1}[f(x)/x]}.$$

On the other hand, since (4) implies $(x^2 + f(x)^2) = 2e^{\pi+\tan^{-1}[f(x)/x]}$, the above right hand side is simplifies into

$$\frac{1}{2}(f(x) + x\,f'(x)) - \frac{(x\,f'(x) - f(x)}{2} == \frac{1}{2}(f(x) + x\,f'(x)) - \frac{1}{2}(x\,f'(x) - f(x)) = f(x)$$

$$F(x) = \frac{x\,f(x)}{2} - e^{\pi+\tan^{-1}[f(x)/x]} + C,$$

**Application 1** Consider the problem of integrating the function

$$y = f(x) = \sqrt{a^2 - x^2}\ , 0 \leq |x| < a\ .$$

Integrating this function is usually accomplished by the conventional substitution $x = a\cos\theta$ or $x = a\sin\theta, but$ here the point being that an application of the Theorem provides a more clearcut method to integrate $y = \sqrt{a^2 - x^2}$. Since the graph of $f(x)$ is simply a semicircle of radius $0 < a$, the obvious polar representation of $y = f(x)$ is $\rho = a$, and therefore $\rho^2(\theta) = a^2$. Since (0, a) is a point on the graph, an application of the Theorem implies that an antiderivative for $y = f(x)$ is,

$$F(x) = \frac{x\,y}{2} - \frac{1}{2}\int_0^{tan^{-1}(\frac{y}{x})} a^2\,\mathrm{d}\theta\ =$$

$$= \frac{x\,y}{2} - \frac{1}{2}\,a^2\theta = \frac{x\,y}{2} - \frac{a^2}{2}\,tan^{-1}\left(\frac{y}{x}\right)$$

$$\frac{x\,y}{2} - \frac{1}{2}\,a^2\theta = \frac{x\,y}{2} - \frac{a^2}{2}\,tan^{-1}\left(\frac{\sqrt{a^2 - x^2}}{x}\right)$$

$$= \frac{x\sqrt{a^2 - x^2}}{2} - \frac{1}{2}\,sin^{-1}(\frac{x}{a})$$

Therefore, the general antiderivative for $y = \sqrt{a^2 - x^2}$ would be

$$F(x) = \frac{x\sqrt{a^2-x^2}}{2} - \frac{1}{2}\,sin^{-1}(\frac{x}{a}) + C.$$