On Sums of Consecutive Complete Squares

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The following theorem shows that infinitely many times it can happen that the sum of squares of a set of consecutive positive integers would be another square number.

Theorem:

(i) For any $n = 1, 2, 3, \ldots$ there are $(6n+1)^2$ consecutive positive integers whose squares add up to a complete square integer.

(ii) For any $n = 2, 3, 4, \ldots$ there are $(6n-1)^2$ consecutive positive integers whose squares add up to a complete square integer.

Note that, in part (ii) when $n = 1$ there are still $(6n-1)^2 = 25$ consecutive integers whose squares add up to a complete square integer, because $0^2 + 1^2 + 2^2 + \ldots + 24^2 = 70^2$, but the first one isn't square of a positive integer. That is why I have started with $n = 2$.

Proof: Shortly, the proof will make use of the following familiar identity, which can easily been established by mathematical induction on the counting number $m = 1, 2, 3, \ldots$.

$$1^2 + 2^2 + \ldots + m^2 = \frac{m(m+1)(2m+1)}{6} \quad . \qquad (1)$$

Because of complete similarity of proofs in the two parts (i) and (ii), I will only show (i), and leave (ii) as an exercise to the reader.

Let $n = 1, 2, 3, \ldots$ and first consider that, since $(6n+1)^2$ is an odd number, and $(6n+1)^2 = 2(18n^2 + 6n) + 1$, any sum of $(6n+1)^2$ consecutive complete square numbers can be expressed as

$$(N - 18n^2 - 6n)^2 + (N - 18n^2 - 6n + 1)^2 + \ldots + (N-1)^2 + N^2 + (N+1)^2$$
$$+ \ldots + (N + 18n^2 + 6n - 1)^2 + (N + 18n^2 + 6n)^2$$

for some $N > 18n^2 + 6n$, if the first term is to be the square of a positive integer. We will show shortly that, with an educated selection of integer N in terms of n, the above sum will be a complete square. To this end, first consider that each pair of symmetric terms $(N-j)^2$ and $(N+j)^2$ on the two sides of N^2 in above sum add up to

$$(N-j)^2 + (N+j)^2 = 2N^2 + 2j^2,$$

Therefore the above sum can be expressed as

$$[\,2\,(18n^2+6n\,)+1\,]\,N^2+2\;[\,1^2+2^2+...+(18n^2+6n)^2\,]\,.$$

Next, using (1) to simplify square bracket (with $m=18n^2+6n$), the above sum becomes,

$$=\;[\,2\,(18n^2+6n\,)+1\,]\,N^2+\;2\;\times\frac{(18n^2+6n)((18n^2+6n+1)(36n^2+12n+1)}{6}\,.$$

Or,

$$(6n+1)^2\;N^2+\frac{(18n^2+6n)((18n^2+6n+1)(6n+1)^2}{3}$$

$$=(6n+1)^2\,[\,N^2+\frac{(18n^2+6n)((18n^2+6n+1)}{3}\,] \qquad .\ (2)$$

$$=(6n+1)^2\,[\,N^2+(6n^2+2n)((18n^2+6n+1)\,]$$

$$=(6n+1)^2\,[\,N^2+2n\,(3n+1)((18n^2+6n+1)\,]$$

As the reader might now have realized, it is enough to express the integer N in terms of n in such a way that inside the above square bracket becomes a complete square; and indeed this is possible if we set,

$$N=\frac{n\,(3n+1)(18n^2+6n+1)}{2}-1\,.$$

Substituting this above N in terms of n inside the last square bracket we get,

$$N^2+2n\,(3n+1)((18n^2+6n+1)$$

$$=[\frac{\mathrm{n}(3n+1)(18n^2+6n+1)}{2}-1]^2+2n\,(3n+1)((18n^2+6n+1)$$

$$=[\frac{n\,(3n+1)(18n^2+6n+1)}{2}]^2-2\times\frac{n\,(3n+1)(18n^2+6n+1)}{2}\;+1\;+2n\,(3n+1)((18n^2+6n+1)$$

$$=[\frac{n\,(3n+1)(18n^2+6n+1)}{2}]^2-n\,(3n+1)((18n^2+6n+1)\;+1\;+2n\,(3n+1)((18n^2+6n+1)$$

$$=[\frac{n\,(3n+1)(18n^2+6n+1)}{2}]^2+n\,(3n+1)((18n^2+6n+1)\;+1$$

$$=[\frac{n\,(3n+1)(18n^2+6n+1)}{2}]^2+2\times\frac{n\,(3n+1)(18n^2+6n+1)}{2}\;+1$$

$$=[\frac{n\,(3n+1)(18n^2+6n+1)}{2}+1]^2$$

This completes the proof. [Note that one of the integers n or else $3n+1$ will always be even, so that we do have a positive integer inside square bracket.

Corollary :

(i) For any $n = 1, 2, 3, \ldots$, and for $N = \frac{n\,(3n+1)(18n^2+6n+1)}{2} - 1$, the following will express the sum of $(6n+1)^2$ complete square integers as another complete integer:

$$(N-18n^2-6n)^2 + (N-18n^2-6n+1)^2 + \ldots + (N-1)^2 + N^2 + (N+1)^2$$
$$+ \ldots + (N+18n^2+6n-1)^2 + (N+18n^2+6n)^2 \quad .$$
$$= (6n+1)^2\,[\frac{n\,(3n+1)(18n^2+6n+1)}{2} + 1]^2$$

(ii) For any $n = 2, 3, \ldots$ l, and for $N = \frac{n\,(3n-1)(18n^2-6n+1)}{2} - 1$, the following will express the sum of $(6n-1)^2$ complete square integers as another complete integer:

$$(N-18n^2+6n)^2 + (N-18n^2+6n+1)^2 + \ldots + (N-1)^2 + N^2 + (N+1)^2$$
$$+ \ldots + (N+18n^2-6n-1)^2 + (N+18n^2-6n)^2 \quad .$$
$$= (6n-1)^2\,[\frac{n\,(3n-1)(18n^2-6n+1)}{2} + 1]^2$$

Examples

(a) For $n = 1$, $(6n+1)^2 = 49$. Therefore by part (i) of the Corollary, setting $N = \frac{1\,(3+1)(18+6+1)}{2} - 1 = 49$,we have

$$(49-24)^2 + (49-23)^2 + \ldots + 49^2 + \ldots + (49+23)^2 + (49+24)^2 =$$
$$49^2\,[\frac{1\,(3+1)(18n+6+1)}{2} + 1]^2 = 49^2 \times 51^2$$

Or, $25^2 + 26^2 + \ldots + 49^2 + \ldots + 72^2 + 73^2 = (49 \times 51)^2$.

(b) For $n = 2$, $(6n-1)^2 = 121$ .Therefore by part (ii) of the Corollary, setting

$N = \frac{2\,(6-1)(72-12+1)}{2} - 1 = 304$, we have

$$(304-60)^2 + (304-59)^2 + \ldots + 304^2 + \ldots + (304+59)^2 + (304+60)^2 == 11^2 \times 306^2.$$

Or, $244^2 + 245^2 + \ldots + 304^2 + \ldots + 363^2 + 364^2 = (11 \times 306)^2$.

(c) For $n = 2$, $(6n+1)^2 = 169$ .Therefore by part (i) of the Corollary, setting

$N = \frac{2\,(6+1)(72+12+1)}{2} - 1 = 594$, we have

$$(594-84)^2 + (594-83)^2 + \ldots + 594^2 + \ldots + (594+83)^2 + (594+84)^2 == 13^2 \times 596^2 .$$

Or, $$510^2 + 511^2 + \ldots + 594^2 + \ldots + 677^2 + 678^2 == (13 \times 596)^2 .$$

(d) For $n = 3$, $(6n-1)^2 = 289$ .Therefore by part (ii) of the Corollary, setting

$N = \frac{3\,(9-1)(162-18+1)}{2} - 1 = 1739$, we have

$$(1739-144)^2 + \ldots + 1739^2 + \ldots + (1739+144)^2 == 117^2 \times 1741^2 .$$

Or, $$1595^2 + 1596^2 + \ldots + 1739^2 + \ldots + 1882^2 + 1883^2 = (17 \times 1741)^2 .$$