<u>A New Geometric Characterization of Parabolas (2025)</u>

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Theorem: [A. Astaneh 2025] Let y = $f(x)$ be a nonlinear entirely differentiable function over the real numbers. Then $f(x)$ is a quadratic function representing a parabola if and only if for any real number x any positive number δ, the slope m = $f'(x)$ of the tangent line at the point P($x, f(x)$ on the graph of the function is given by

$$f'(x) = \frac{f(x+\delta)-f(x-\delta)}{2\delta}. \qquad (*)$$

Note: In what follows we will to this as property (*).

Proof : Let us first assume that y = $f(x)$ is a quadratic function representing a parabola. That is, $f(x) = a\ x^2 + bx + c$. Then on one hand we know that $f'(x) = 2a\,x + b$. On the other hand for δ the right hand side in the fraction becomes

$$\frac{f(x+\delta)-f(x-\delta)}{2\delta} = \frac{a(x+\ \delta)^2 + b(x+\ \delta)) + c - a(x-\ \delta)^2 - b(x-\ \delta) - c}{2\delta}$$

$$= \frac{ax^2 + 2a\delta x + a\,\delta^2 + bx + b\,\delta\ + c - ax^2 + 2a\delta x - a\,\delta^2 - bx + b - c}{2\delta}$$

$$= \frac{4a\delta x + 2b\,\delta}{2\delta} = 2ax + b = f'(x)$$

This proved the only if part.

To prove the more challenging if part, we need to quote one of the two already known characterizations for parabolas of mine published in Issue #15, 2011 of the Pi in the Sky magazine, and for completeness of the present article, here we first bring that particular characterizations as a proposition.

Proposition: (see Pages 16, Issue #15 of π in the Sky for proof)
Let $f(x)$ be a **nonlinear** differentiable function over the real line. Then $f(x)$ is a quadratic function if and only if for any two distinct points $A((a, f(a))$ and $B(b, f(b))$ on the graph of $f(x)$ the slope of the secant segment AB supported by the graph of $f(x)$ is equal to the average between the slopes of the tangent lines T_a and T_b at $((a, f(a))$ and $B(b, f(b))$ respectively. In algebraic form this means for any two points $A((a, f(a))$ and $B(b, f(b))$ on the graph of $f(x)$ the following relation holds.

$$\frac{f(b)-f(a)}{b-a} = \frac{f\prime(b)+f\prime(a)}{2} \ . \qquad (**)$$

We now bring the proof of the if part of the Theorem.

Let $f(x)$ be a nonlinear entirely differentiable function, and assume that for every real number x, and every $\delta > 0$ the property (*) holds. Then Upon cross multiplication of the relation (*) we get,

$$f(x+\delta) - f(x-\delta) = 2\delta f'(x). \qquad (1)$$

Since (2) holds for all real numbers x and all $\delta > 0$, we will momentarily keep the real number x fixed, and differentiate both sided of (1) with respect to the variable δ. Then the result can be expressed as

$$\frac{f\prime(x+\delta)+f\prime(x-\delta)}{2} = f'(x) \ .$$

If we now substitute the fraction $\frac{f(x+\delta)-f(x-\delta)}{2\delta}$ for $f'(x)$ from the relation (*) in the latter equation we get,

$$\frac{f\prime(x+\delta)+f\prime(x-\delta)}{2} = \frac{f(x+\delta)-f(x-\delta)}{2\delta} \qquad (2)$$

We now let x vary over all real numbers, so that (3) holds for all real numbers x and any all $\delta > 0$. Now, since we are about to apply the above Proposition 2, let a and b be arbitrary real numbers, say $a < b$, and set $b - a = 2\delta$, $x = \frac{a+b}{2}$. Then relation (2) can be rewritten in terms of a and b as follows

$$\frac{f(b)-f(a)}{b-a} = \frac{f\prime(b)+f\prime(a)}{2} \ . \qquad (3)$$

Hence, we conclude that for any two real numbers a and b the relation (3) holds. But the relation (3) is the same as the relation (**) in the above Proposition. Hence, by above Proposition, $f(x)$ has to be a quadratic function and the proof of the Theorem is complete.

Speaking of new in the tile of the article, we also conclude,

<u>A New Poetic Characterization of Parabolas:</u> Since the characterization of parabolas we have just proved, is that for every real number x and every positive number δ, the relation

$$f'(x) = \frac{f(x+\delta)-f(x-\delta)}{2\delta} \ . \qquad (*)$$

holds, again upon substitution $b - a = 2\delta$, $x = \frac{a+b}{2}$ the relation (*) can be expressed as

$$f'\left(\frac{a+b}{2}\right) = \frac{f(b)-f(a)}{b-a}.$$

Now substituting $\frac{f\prime(b)+f\prime(a)}{2}$ for $\frac{f(b)-f(a)}{b-a}$ in the above relation we get

$$\frac{f'(a)+f'(b)}{2} = f'\left(\frac{a+b}{2}\right).$$

The above unique relation for parabolas that holds for all real numbers a and b allows us to express the following poetic characterization:

Parabolas are the only existing curves for which we can claim the following mathematical rhyme for them;

The “average” between the “derivatives” of a function representing a parabola at any two distinct real numbers is the “derivative” of the function at the “average” between the two distinct real numbers!