Pythagorean Triangles; a complete description of Pythagorean Twins

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Recall that a Pythagorean Triangle is a right triangle whose three sides are all integers.
The main objective of this article is to present a complete description of all those Pythagorean Triangles, for which the difference between the hypotenuse and the longer leg of the right angle is exactly one unit. The description is presented by means of simple sequence available to any BC secondary student of grade 10 and above.

Definition: I will refer to a pair of consecutive positive integers $(H-1, H)$ as "Pythagorean Twins" if the difference of the square of the consecutive numbers $H^2-(H-1)^2=2H-1$ is a complete square integer.

A particular subsequence of the sequence of Pythagorean Triangles obtained from Pythagorean Twins reveals a mathematically amusing geometrical pattern about this particular class of Pythagorean Triangles. More precisely if h_k is the hypotenuse of the Pythagorean Triangle T_k obtained from Pythagorean Twins (h_k-1, h_k), then the hypotenuse of T_k becomes the shortest side of the Pythagorean Triangle T_{k+1} obtained from the next pair $(h_{k+1}-1, h_{k+1})$.

We first note that for any Pythagorean Twins $(H-1, H)$, the radical $\sqrt{2H-1}$ being an integer, the triangle with sides $H, H-1$, and $\sqrt{2H-1}$ will be a Pythagorean Triangle.
Also having in mind that the integer $2H-1$ is both odd and at the same time a complete square, it is easy to conclude that H must be an odd integer. This is because knowing that $2H-1$ is square of an odd number, say of the form $2l+1$ where l is another integer, we will have $2H-1=(2l+1)^2=4l^2+4l+1$, from which it follows $H=2l^2+2l+1$ and this represents an odd number.

Now I bring the main proposition of the article, which describes all existing Pythagorean Twins.

Proposition: The following sequence describes all existing Pythagorean Twins $(H-1, H)$.

$H_k=2k^2+2k+1, \quad k=1,2,3,\ldots$.

Proof ; simply follows from the identity

$$(2k^2+2k+1)^2-(2k^2+2k)^2=(2k+1)^2.$$

Note that, this also means the gap between any pair of Pythagorean Twins (H_k-1, H_k) and the next pair $(H_{k+1}-1, H_{k+1})$ is simply

$$2(k+1)^2+2(k+1)+1-(2k^2+2k+1)=4(k+1).$$

Corollary All existing Pythagorean Triangles obtained from Pythagorean Twins must have the following sides; in the decreasing order of the three sides .

$$2k^2+2k+1, \; 2k^2+2k, \text{ and } 2k+1$$

for $k=1,2,3,\ldots$.

The following list in blue shows the first one hundred hypotenuses for Pythagorean Triangles obtained from first 100 Pythagorean Twins. As an example how to obtain the Pythagorean Triangle corresponding to the hypotenuse, say $H = 13$ in the list , just consider $H = 13$ and $H - 1 = 12$ to be the hypotenuse and the middle side of a Pythagorean Triangle. Then, since $13^2 - 12^2 = 5^2$, the shorter leg of the right angle in the triangle would be would be 5.

Also note, an interesting feature of the following list (as supported by the Proposition), that the entries 221, 20201, 2002001, and 200020001, … (last two showing up eventually) form the hypotenuse of the 10th, the100th, the 1000th, and 10000th , … of the existing Pythagorean Triangles obtained fron Pythagorean Twins .

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> seq(2*(k^2)+2*k+1,k=1..100);
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5, 13, 25, 41, 61, 85, 113, 145, 181, 221, 265, 313, 365, 421, 481, 545, 613, 685, 761, 841, 925, 1013, 1105, 1201, 1301, 1405, 1513, 1625, 1741, 1861, 1985, 2113, 2245, 2381, 2521, 2665, 2813, 2965, 3121, 3281, 3445, 3613, 3785, 3961, 4141, 4325, 4513, 4705, 4901, 5101, 5305, 5513, 5725, 5941, 6161, 6385, 6613, 6845, 7081, 7321, 7565, 7813, 8065, 8321, 8581, 8845, 9113, 9385, 9661, 9941, 10225, 10513, 10805, 11101, 11401, 11705, 12013, 12325, 12641, 12961, 13285, 13613, 13945, 14281, 14621, 14965, 15313, 15665, 16021, 16381, 16745, 17113, 17485, 17861, 18241, 18625, 19013, 19405, 19801, 20201

Remark: A specific subsequence of the above sequence which is inductively defined by

$$h_1 = 5, \quad h_{k+1} = (h_k^2 + 1)/2, \quad k = 1, 2, 3, \dots \quad ,$$

with the first four members being $h_1 = 5$, $h_2 = 13$, $h_3 = 85$, $h_4 = 3613, \dots$ has the interesting property that ; the hypotenuse of the Pythagorean Triangle T_k created by the pair of integers $(h_k, h_k - 1)$, becomes the shortest side of the Pythagorean Triangle T_{k+1} created by the pair of integers $(h_{k+1}, h_{k+1} - 1)$.

Note that, since $h_{k+1}^2 - (h_{k+1} - 1)^2 = 2h_{k+1} - 1 = 2\left[\frac{h_k^2+1}{2}\right] - 1 = h_k^2$, the pair of integers defined by h_{k+1} and $h_{k+1} - 1$ are indeed Pythagorean Twins, and therefore each h_{k+1} defined by the above recursive formula does belong to the above list.