Binomial Expansion for Reciprocal Powers

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In the previous article of this same section of the website I showed how Binomial Expansion looks like for $(a+b)^{-n}$ when n is a positive integer. In the present article I will show how the expansion will look like for $(a+b)^{\frac{1}{n}} = \sqrt[n]{a+b}$, where n is a positive integer. Again I start with a lemma first, but this time mathematical induction will not work to prove the lemma. What will work is a simple application of the so called Maclaurin series for function $\sqrt[n]{1+x} = (1+x)^{\frac{1}{n}}$.

Lemma For any positive integer $n > 1$, and any real number $-1 < x < 1$,

$$\sqrt[n]{1+x} = 1 + \frac{1}{n}\frac{x}{1!} + \frac{1}{n}\left(\frac{1}{n}-1\right)\frac{x^2}{2!} + \frac{1}{n}\left(\frac{1}{n}-1\right)\left(\frac{1}{n}-2\right)\frac{x^3}{3!} + \cdots \quad (1)$$

In sigma notation this means,

$$\sqrt[n]{1+x} = 1 + \sum_{k=1}^{\infty} \frac{1}{n}\left(\frac{1}{n}-1\right)\left(\frac{1}{n}-2\right)\ldots\left(\frac{1}{n}-k+1\right)\frac{x^k}{k!}$$

Proof: I recall that the Maclaurin series (that is the Taylor series about the origin $a = 0$) for any infinitely differentiable function $f(x)$ is

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ \ldots$$

If you now apply the above to the function $f(x) = \sqrt[n]{1+x} = (1+x)^{\frac{1}{n}}$ over the interval $-1 < x < 1$ you exactly arrive at the equation (1). This completes the proof of the Lemma.

Remark : The above Lemma also implies the rare occurrence of the fact that for $-1 < x < 1$,

$$\left[1 + \sum_{k=1}^{\infty} \frac{1}{n}\left(\frac{1}{n}-1\right)\left(\frac{1}{n}-2\right)\ldots\left(\frac{1}{n}-k+1\right)\frac{x^k}{k!}\right]^n = 1+x$$

$$= \frac{1}{1 - x + x^2 - x^3 + \ \ldots}.$$

I now bring a generalization of the above Lemma as a proposition.

Proposition : Let $a, b > 0$ be real numbers and let $-\frac{b}{a} < x < \frac{b}{a}$. Then for any positive integer $n > 1$ we have the following expansion,

$$\sqrt[n]{ax+b} = \sqrt[n]{b} + \sum_{k=1}^{\infty} \frac{1}{n}\left(\frac{1}{n}-1\right)\left(\frac{1}{n}-2\right)\ldots\left(\frac{1}{n}-k+1\right)\frac{a^k x^k}{k!\, b^{k-\frac{1}{n}}} .$$

Proof: Since $\sqrt[n]{ax+b} = \sqrt[n]{b}\sqrt[n]{1+\frac{ax}{b}}$ and since $-\frac{b}{a} < x < \frac{b}{a}$ implies $-1 < ax/b < 1$, the above Lemma (when x is replaced by (ax/b)) leads to

$$\begin{aligned}\sqrt[n]{ax+b} &= \sqrt[n]{b}\ \sqrt[n]{1+\frac{ax}{b}} \\ &= \sqrt[n]{b}\ \left[1+\sum_{k=1}^{\infty} \frac{1}{n}\left(\frac{1}{n}-1\right)\left(\frac{1}{n}-2\right)\ldots(\frac{1}{n}-k+1)\frac{a^k x^k}{k!\, b^k}\right] \\ &= \sqrt[n]{b} \\ &+ \sum_{k=1}^{\infty} \frac{1}{n}\left(\frac{1}{n}-1\right)\left(\frac{1}{n}-2\right)\ldots\left(\frac{1}{n}-k+1\right)\frac{a^k x^k}{k!\, b^{k-\frac{1}{n}}}\end{aligned}$$

and the proof is complete.

<u>Corollary:</u> <u>Binomial Expansion for Reciprocal Powers</u>

Let $b > a > 0$ be real numbers and let $n > 1$ be a positive integer , then if you set $x = 1$ in the assertion of the above Proposition, you get the following binomial expansion for reciprocal described as :

$$\sqrt[n]{a+b} = \sqrt[n]{b} + \sum_{k=1}^{\infty} \frac{1}{n}\left(\frac{1}{n}-1\right)\left(\frac{1}{n}-2\right)\ldots\left(\frac{1}{n}-k+1\right)\frac{a^k}{k!\, b^{k-\frac{1}{n}}} .$$

Or, in shorter notation as

$$\sqrt[n]{a+b} = \sqrt[n]{b} + \sum_{k=1}^{\infty}\prod_{j=0}^{k-1}\left(\frac{1}{n}-j\right)\ \frac{a^k}{k!\, b^{k-\frac{1}{n}}} .$$