A Method of Implicit Integration

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Abstract

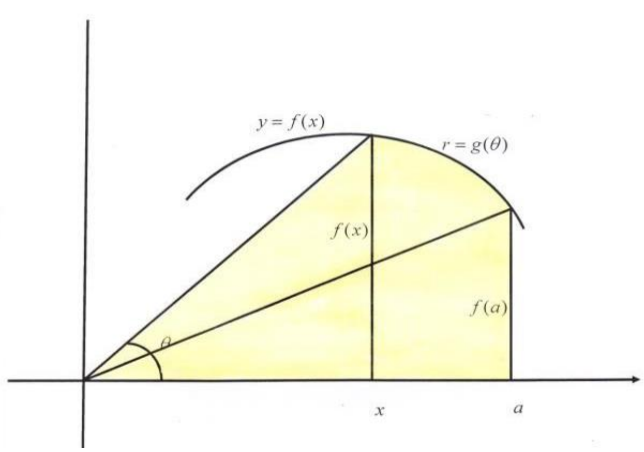
In this article we introduce a method that enable us to find antiderivatives for differentiable functions defined through a specific class of implicit relations $R\ (x, y)$= 0 on the xy- coordinate system. The method works for the class of relations for which upon polar substitutions $x = \rho\ cos\ \theta$ and $y = \ \rho\ sin\ \theta$ the resulting polar relation P (ρ, θ)= 0 can be solved for the variable ρ in terms of θ and that the trigonometric function $\rho^2(\theta)$ is in integrable in terms of θ.

As far as the author's previous decades of teaching Calculus at all levels is concerned, the method might be the closest one can get to integrate an implicitly defined function, yet. The author came up with the method early 2016 while teaching AP-Calculus at Prince of Wales Secondary in Vancouver, and a posted a short and account of it on the BCAMT list serve the same year t. The method has not been mentioned in any classic Calculus textbooks the author has seen, so it might sound reasonable to assume that the method is has not been explored before.

Here, we first start by stating the main Theorem introducing the method, and then bring 3 Example to show how the method works for concrete relations $R\ (x, y)$= 0.

Theorem [A. Astaneh] Let $y = f(x)$ be a real valued continuous function defined through an implicit relation $R\ (x, y)$= 0, and let P (ρ, θ)= 0 be the polar version of the elation obtained upon substitution of $x = \rho\ cos\ \theta$ and $y = \ \rho\ sin\ \theta$ in the relation $R\ (x, y)$= 0. Furthermore, assume that the latter polar relation can be solved for ρ in terms of θ*, and that the* trigonometric function $\rho\ (\theta)^2$ is integrable in terms of θ. Then for any real number a in the domain of $f(x)$, the following provides an antiderivative for $f(x)$,

$$F(x) = \frac{x\, f(x)}{2} - \frac{1}{2}\int_{tan^{-1}(\frac{f(a)}{a})}^{tan^{-1}(\frac{f(x)}{x})} \rho\ (\theta)\ ^2 d\theta\ \ + C.$$



Note that, considering the relations $sin\ \theta = \frac{y}{\sqrt{x^2+y^2}}$, $cos\ \theta = \frac{x}{\sqrt{x^2+y^2}}$, and $\theta = tan^{-1}(\frac{y}{x})$, the function $F(x)$ will eventually be expressed in terms of general x and y.

Proof: We first show how the formula (1) comes about as an antiderivative for $f(x)$, and then use the Leibniz Integral Rule to bring a proof independent from above figure to reaffirm that $F(x)$ define by (1) is an antiderivative for $y = f(x)$. To this end, let a be a real number in the domain of $f(x)$ defined by the elation R (x, y) = 0. Then for any given x other than a in the domain of $f(x)$ consider the colored area enclosed by the x −axis, the vertical line $x = a$, the graph of $f(x)$, and by the ray of line emanating from the origin and making an angle of $\theta = tan^{-1}(\frac{f(x)}{x})$ with the positive x −axis. Now let $F(x)$ be the antiderivative of $f(x)$ defining the area under graph of $f(x)$ and directly above the interval [x,a]. Also let A_θ denote the polar area between the polar curve r $= \rho\ (\theta)$ lying between the rays of lines defined by angles $\theta_1 = tan^{-1}(\frac{f(a)}{a})$ and $\theta = tan^{-1}(\frac{f(x)}{x})$. Then, since the area of right triangle with the height $f(x)$ seen above $\frac{x\, f(x)}{2}$, the described enclosed area can be represented by the two equal expressions $\frac{x\, f(x)}{2} + F(x) = \frac{a\, f(a)}{2} + A_p$,
Now considering the antiderivative $F(x)$ of $f(x)$, and applying the Fundamental Theorem of Calculus for the area under graph $f(x)$ and over the interval [x,a], as well as the wellknown polar area formula $\frac{1}{2}\int_{\theta_1}^{\theta} \rho\ (\theta)\ ^2 d\theta$ for the polar area A_p, the above relation can be expressed as

$$\frac{x\, f(x)}{2} + F(a) - F(x) = \frac{a\, f(a)}{2} + \frac{1}{2}\int_{tan^{-1}(\frac{f(a)}{a})}^{tan^{-1}(\frac{f(x)}{x})} \rho\ (\theta)\ ^2 d\theta.$$

Therefore, ignoring the constant numbers in the above equations we conclude that

$$F(x) = \frac{x\, f(x)}{2} - \frac{1}{2}\int_{tan^{-1}(\frac{f(a)}{a})}^{tan^{-1}(\frac{f(x)}{x})} \rho\ (\theta)\ ^2 d\theta\ ,$$

Is an antiderivative for $f(x)$. Since the above argument relies heavily on above diagram, one may ask for a proof that $F(x)$ defined by (1) above is indeed an antiderivative for (x) .
We now use the Leibniz Integral Rule (the fundamental theorem of calculus combined with the chain rule) to reassure that $F'(x) = f(x) = y$, as follows,

$$F\ '(x) = \frac{d}{dx}[\frac{xy}{2} - \frac{1}{2}\int_{tan^{-1}(\frac{f(a)}{a})}^{tan^{-1}(\frac{y)}{x})} \rho\ (\theta)\ ^2 d\theta\ ] =$$

$$\frac{y + xy'}{2} - \frac{1}{2}\left[\rho\ \left(tan^{-1}(\frac{y}{x})\right)^2 \frac{d}{dx}(tan^{-1}\left(\frac{y)}{x}\right))\right] =$$

$$= \frac{y + xy'}{2} - \frac{1}{2}\left[\rho\ (\theta)^2 \frac{d}{dx}(tan^{-1}\left(\frac{y)}{x}\right))\right] =$$

$$\frac{y + xy'}{2} - \frac{1}{2}\left[(\ x^2 + y^2) \times \frac{xy' - y}{x^2 + y^2}\right] = \frac{y + xy'}{2} - \frac{1}{2}(xy' - y) = y = f(x).$$

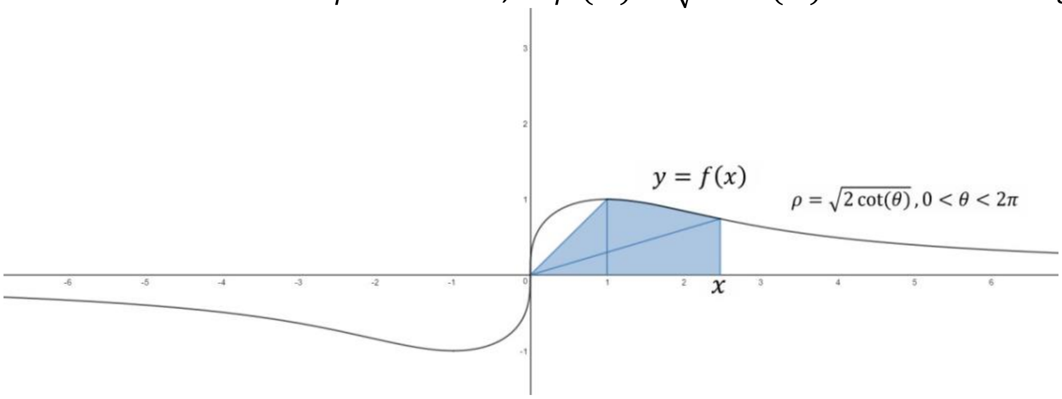
This completes the proof of the Theorem.

Example 1 Let $y = f(x)$ be the function implicitly defined by the relation

$$y(x^2 + y^2) = 2x\,, \quad x \geq 0, y \geq 0. \qquad (1)$$

Find an anti-derivative $F(x)$ for $f(x)$ with initial condition $F(1) = 0$.

Analytic Solution: Upon polar substitutions $x = \rho \cos\theta$ and $y = \rho \sin\theta$ in relation (1), the relation is converted into $\rho^{\,2} = 2\cos\theta$, or $\rho\,(\theta) = \sqrt{2\cos(\theta)}$ with the following polar graph



Now let $F(x)$ be the antiderivative of $f(x)$, defining the area under the graph directly above the interval $[1, x]$, and let A_x denote the same area. Also let A_θ denote the polar area enclosed by the curve r $= \rho\,(\theta)$ and the rays of lines defined by angles $\frac{\pi}{4}$ (note that the point $(1,1)$ is on the curve) and $\theta = tan^{-1}[\frac{f(x)}{x}]$. Then the above figure shows the validity of the relation between the two areas A_x and A_θ,

$$\frac{1\,f(1)}{2} + A_x = \frac{x\,f(x)}{2} + A_\theta.$$

Now, by Fundamental Theorem of Calculus, and the polar formula for the area A_θ, the above relation implies

$$\frac{1}{2} + F(x) - F(1) = \frac{x\,f(x)}{2} + \frac{1}{2}\int_{tan^{-1}[\frac{f(a)}{a}]}^{\frac{\pi}{4}} \rho\,(\theta)^{\,2} d\theta,$$

$$F(x) = \frac{x\,f(x)}{2} - \frac{1}{2} + \frac{1}{2}\int_{tan^{-1}\{\frac{f(a)}{a}]}^{\frac{\pi}{4}} 2\cot(\theta)\,d\theta,$$

$$F(x) = \frac{x\,f(x)}{2} - \frac{1}{2} + ln(sin(\theta)\Bigg|_{tan^{-1}[\frac{f(a)}{a}]}^{\frac{\pi}{4}}$$

$$F(x) = \frac{x\,f(x)}{2} - \frac{1}{2} - \frac{1}{2}\,ln(2) - ln\frac{y}{\sqrt{x^2+y^2}}.$$

And therefore,

$$F(x) = \frac{x\,f(x)}{2} + ln\frac{\sqrt{x^2+f(x)^2}}{\sqrt{2}\,f(x)} - \frac{1}{2}$$

is the antiderivative of the function $y = f(x)$ implicitly defined by relation (1), with initial condition $F(1) = 0$.

Algebraic Verification of the Solution: It is enough to show that the derivative of $F(x)$ as expressed above is $y = f(x)$. To this end, in the following manipulation we will make use of the original implicit relation (1), in the sense that when a denominator $y(x^2 + y^2)$ show up in the process we will replace it with $2x$.

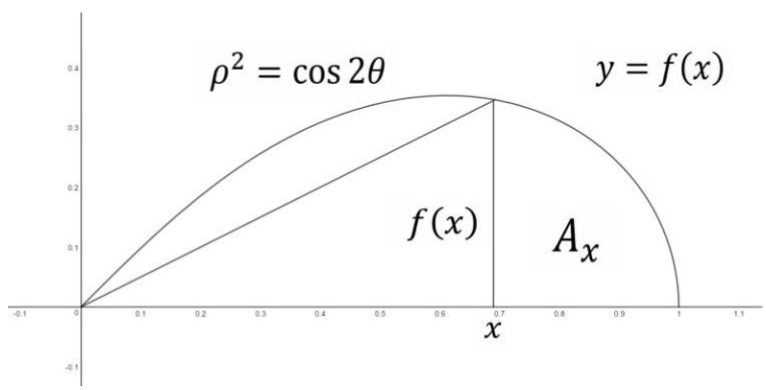
$$F'(x) = \frac{d}{dx}\left[\frac{xy}{2} + \ ln\frac{\sqrt{x^2+y^2}}{\sqrt{2}\,y} - \frac{1}{2}\right] =$$
$$\frac{d}{dx}\left[\frac{xy}{2} + + \frac{1}{2} ln(x^2 + y^2) - ln\ 2\,\right] =$$
$$\frac{y + xy'}{2} + \frac{2x + 2yy'}{2(\,x^2 + y^2)} - \frac{y'}{y} = \frac{y + xy'}{2} + \frac{y(x + yy') - y'(\,x^2 + y^2)}{y(\,x^2 + y^2)}$$
$$\frac{y + xy'}{2} + \frac{y(x + yy') - y'(\,x^2 + y^2)}{2x} =$$
$$\frac{xy + \ x^2y' + xy + y^2y' - x^2y' - y^2y'}{2x} = \ y = f\ (x).$$

Example 2 Let $y = f(x)$ be the function implicitly defined by the relation

$$(x^2 + y^2)^2 = \ x^2 - y^2, \quad 0 \le x \le 1,\ 0\ \le y\ . \ (2)$$

Find an anti-derivative $F(x)$ for $f(x)$ with initial condition $F(1) = 0$.

Analytic Solution: Substituting $x = \rho\ cos\ \theta$ and $y = \ \rho\ sin\ \theta$ in relation (2), the Cartesian relation is converted into a polar relation as $\rho^{\,4} = \rho^{\,2}\ cos^2\theta - \rho^{\,2}\ sin^2\theta = \rho^{\,2}\ cos\ 2\theta,$ or $\rho^{\,2} = cos\ 2\theta,$ with the following graph



Since from relation (2) we know that the graph goes through the point $(1{,}0)$, let $F(x)$ be the antiderivative of $f(x)$ defined by $F(x)\ = A_x = \int_x^1 f(x)\,dx,$ so that $F(x)$ would be the area under the curve directly above the interval $[x, 1]$, and $F(1) = 0$, and also let A_θ = $\frac{1}{2}\int_0^{tan^{-1}[\frac{f(x)}{x}]} \rho\ (\theta)^{\,2} d\theta$ be the polar area enclose by the curve and the rays of lines $\theta = 0\ , \theta = tan^{-1}[\frac{f(x)}{x}]$. Then as seen in the diagram,

$$A_x = A_\theta - \ \frac{x\,f(x)}{2}.$$

Now, by the Fundamental Theorem of Calculus,

$$A_x = F(1) - F(x) = \frac{1}{2}\int_0^{tan^{-1}[\frac{f(x)}{x}]} cos\,2\,\theta\,d\theta \; - \; \frac{x\,f(x)}{2}.$$

Hence, considering $F(1) = \frac{x\,f(x)}{2} - \frac{1}{4}[sin(2\theta]\Big|_0^{tan^{-1}[\frac{f(x)}{x}]}$

Or,

$$F(x) \;=\; \frac{x\,f(x)}{2} - \frac{x\,f(x)}{2[\,x^2 + f(x)^2]}\;.$$

Therefore, the above $F(x)$ is the antiderivative of the implicitly defined function by relation (2) satisfying $F(1) = 0$.

Algebraic Verification of the Solution: Again, we will show the derivative of $F(x)$ as expressed above is $y = f(x)$. To this end, again at some point in the process of the manipulation bellow, when the expression $(x^2 + y^2)^2$ shows up as a denominator will make use of the implicit relation (2), and replace the expression by $(x^2 - y^2)$.

$$F\,'(x) = \frac{d}{dx}[\frac{xy}{2} - \frac{x\,y}{2(\,x^2+y^2)}] =$$

$$\frac{y + xy'}{2} - \frac{1}{2}\,\frac{(y + xy')(\,x^2 + y^2) - xy(2x + 2yy')}{(x^2 + y^2)^2} \;=$$

$$\frac{y + xy'}{2} - \frac{xy'\,(x^2 - y^2) - y(\,x^2 - y^2)}{2\,(x^2 - y^2)} \;=$$

$$\frac{y+xy\prime}{2} - \frac{xy\prime - y}{2} = y.$$

Remark: Incidentally the implicit relation (2) in Example 2, under the given conditions $0 \le x \le 1,\; 0 \le y$ can be solved for y in terms of x as the (non-handsome!) expression,

$$y = \frac{1}{\sqrt{2}}\,\sqrt{\sqrt{8\,x^2 + 1} - (2\,x^2 + 1)}\;\;,\;\; 0 \le x \le 1.$$

Therefore, indeed the above solution to Example 2 has led to finding the solution to the following rather challenging integral,

$$\int \frac{1}{\sqrt{2}}\,\sqrt{\sqrt{8\,x^2 + 1} - (2\,x^2 + 1)}\;\;dx\,,\;\; 0 \le x \le 1$$

as

$$\int \frac{1}{\sqrt{2}}\,\sqrt{\sqrt{8\,x^2 + 1} - (2\,x^2 + 1)}\;\;dx = \frac{x\,f(x)}{2} - \frac{x\,f(x)}{2[\,x^2+f(x)^2]}\;\; + C,$$

where $f(x) \;= \frac{1}{\sqrt{2}}\,\sqrt{\sqrt{8\,x^2 + 1} - (2\,x^2 + 1)}\,,\;\; 0 \le x \le 1.$

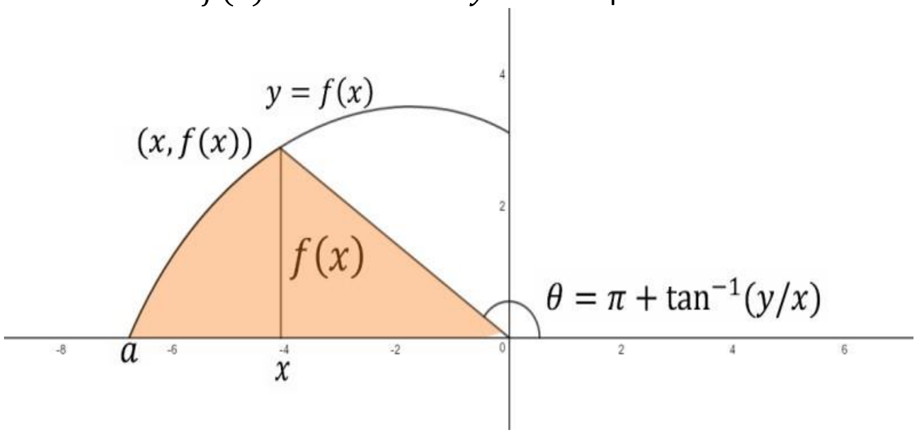
This fact can be considered as a useful technique of integration to find specific class of integrals, as the author for one doesn't recall any integration technique that can integrate the above function where $f(x)$.

Example 3 Find an antiderivative for the function Let $y = f(x)$ implicitly defined by the relation

$$ln(\frac{x^2+y^2}{2}) = \pi + tan^{-1}(\frac{y}{x}),\;\; -\sqrt{2}\,e^{\frac{\pi}{2}} \le x < 0\,, 0 < y < \sqrt{2}\,e^{\frac{3\pi}{4}} \quad (3)$$

Note that under the restricted intervals for variables x and y in (3) the point $(a, 0)$ with

$a = -\sqrt{2}\, e^{\frac{\pi}{2}}$ is the x −intercept of $y = f(x)$, whereas because of the fact that $x = 0$ isn't in the domain of the function $f(x)$ doesn't have a y −intercept.



Analytic Solution: Substituting $x = \rho\, cos\theta$ and y= $\rho\, sin\theta$ in relation (3), and considering the domin and range of the function, the polar angle between positive x-axis and the line segment from the origin to the point (x , f (x)) will be represented by $\theta = \pi + tan^{-1}(\frac{y}{x})$ with $\pi < \theta < \frac{3\pi}{2}$, and the polar version of relation (3) would become $ln(\frac{\rho^2}{2}) = \theta$, or $\rho^2 = 2e^{\theta}$. Now let $F(x)$ be the antiderivative defining the area under the curve that is directly above the interval $[a, x]$. Then, considering that the positive number $-\frac{x\, f(x)}{2}$ represents the area of the right triangle part of the shaded area in the figure above, the total shaded polar area A_θ will be the same as

$$\frac{1}{2}\int_0^{tan^{-1}[\frac{f(x)}{x}]} \rho\,(\theta)\,^2 d\theta \;= F(x) - \frac{x\, f(x)}{2} \;=\frac{1}{2}\int_0^{tan^{-1}[\frac{f(x)}{x}]} 2e^{\,\theta} d\theta = e^{\,\theta}\Bigg|_{\pi + tan^{-1}[\frac{f(x)}{x}]}^{\pi} = e^{\,\pi} - e^{\,\pi + tan^{-1}[\frac{f(x)}{x}]}.$$

Therefore $F(x)$=$\frac{x\, f(x)}{2} - e^{\,\pi + tan^{-1}\left[\frac{f(x)}{x}\right]} + e^{\,\pi}$ will be the required antiderivative of the implicitly defined function $y = f(x)$ by relation (3), under the given restrictions of the variables.

Algebraic Verification of the Solution:

$$F\,'(x) = \frac{d}{dx}[\frac{x\, f(x)}{2} - e^{\,\pi + tan^{-1}[\frac{f(x)}{x}]}\,] =$$

$$\frac{1}{2}[f(x) + xf\,'\,(x)] - \frac{xf\,'(x) - f(x)}{[\,x^2 + f(x)^2]}\; e^{\,\pi + tan^{-1}[\frac{f(x)}{x}]}.$$

Now, since relation (3) implies $[\,x^2 + f(x)^2] = 2\, e^{\,\pi + tan^{-1}[\frac{f(x)}{x}]}$, we conclude that

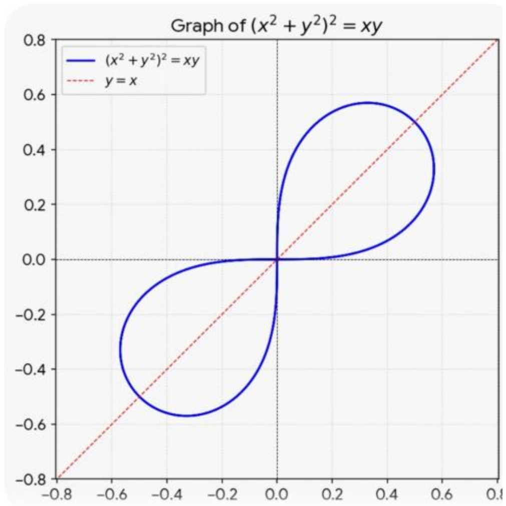
$F\,'(x) = \frac{1}{2}[f(x) + xf\,'\,(x)] - \frac{1}{2}[xf\,'\,(x)\; - f(x)] = f(x)$,

and verification of the solution is complete.

We end the article by presenting two applications of the method under discussion. The first one, like what mentioned in the Remark on page 8 shows that the method can be used as a new technique for integration, and the second one is to use it to find a trigonometric approximation for the function $f(x) = ln\, x$.

Example 4 Find an anti-derivative $F(x)$ for the function $y = f(x)$ implicitly defined by the relation

$$(x^2 + y^2)^2 = xy, \quad -\frac{\sqrt[4]{27}}{4} \leq x \leq \frac{\sqrt[4]{27}}{4}, \; -\frac{\sqrt[4]{27}}{4} \leq y \leq \frac{\sqrt[4]{27}}{4}. \qquad (6)$$



Note, that the above graph is known as the Lemniscate of Bernoulli.

Analytic Solution: Substituting $x = \rho \cos\theta$ and $y = \rho \sin\theta$ in relation (6), the Cartesian relation is converted into a polar relation as $\rho^4 = \rho^2 \sin\theta \cos\theta$, or $\rho^2 = \frac{1}{2}\sin 2\theta$, which also implies $\rho = \sqrt{\frac{1}{2}\sin 2\theta}$. Since graph of the relation (6) passes through the origin, to find an anti-derivative $F(x)$ for $y = f(x)$ we will use the more convenient version

$$F(x) = \frac{x\,y}{2} - \frac{1}{2}\int_0^{\tan^{-1}(\frac{y}{x})} \rho^2(\theta)\, d\theta$$

of the Theorem. Therefore, the antiderivative will amount to,

$$F(x) = \frac{x\,y}{2} - \frac{1}{2}\int_0^{\tan^{-1}\left(\frac{y}{x}\right)} \frac{1}{2}\sin 2\theta\, d\theta = \frac{x\,y}{2} - \frac{1}{2}\int_0^{\tan^{-1}\left(\frac{y}{x}\right)} \sin\theta\;\cos\theta\, d\theta$$

$$= \frac{x\,y}{2} + \frac{1}{4}\cos^2\theta \;\Big|_0^{\tan^{-1}\left(\frac{y}{x}\right)}$$

$$= \frac{x\,y}{2} + \frac{x^2}{4\,(x^2 + y^2)}$$

and hence

$$F(x) = \frac{x\,y}{2} + \frac{x^2}{4(\,(x^2+y^2)}$$

is the required antiderivative for the implicitly defined function $y = f(x)$ by relation (6).

Algebraic Verification of the Solution: We will show the derivative of $F(x)$ as expressed above is $y = f(x)$. And again, to this end, in the following we make use of relation (6) by substituting $(x^2+y^2)^2$ with xy at some point.

$$F\,'(x) = \frac{d}{dx}\left[\frac{xy}{2} + \frac{x^2}{4(\,x^2+y^2)}\right] =$$

$$\frac{y+xy\prime}{2} + \frac{1}{4}\,\frac{2x(\,x^2+y^2) - x^2(2x+2yy\prime)}{(x^2+y^2)^2} =$$

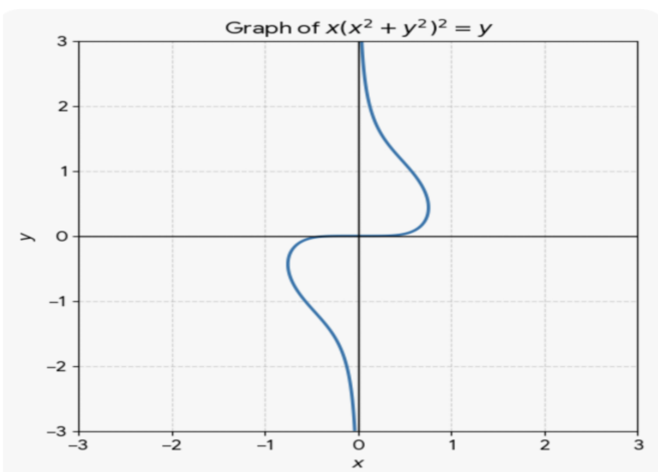
$$\frac{y+xy\prime}{2} + \frac{2xy^2-2x^2yy\prime}{4xy} = \frac{y+xy\prime}{2} + \frac{y-xy\prime}{2} = y.$$

Therefore $F\,'(x) = y = f(x)$, and the verification is established.

Indeed, Example 4 can be generalized to an infinite sequence of Relations to which the Method can be applied successfully, as seen in our next Example,

Example 6 Let $y = f(x)$ be the function implicitly defined by the relation

$$x\,(x^2+y^2) = y, \qquad 0 \le |x| \le \tfrac{1}{\sqrt{2}},\ 0 \le |y| \le \tfrac{1}{\sqrt{2}}. \qquad (7)$$



Find an anti-derivative $F(x)$ for $f(x)$.

Analytic Solution: Substituting $x = \rho\, cos\, \theta$ and $y = \rho\, sin\, \theta$ in relation (7), the Cartesian relation is converted into a polar relation as $\rho^{\,2} = tan\,\theta$.
Again, considering that the point with the Cartesian coordinates $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ is on the graph of relation (5), to find an anti-derivative for $f(x)$ it is enough to use the version

$$F(x) = \frac{x\,f(x)}{2} - \frac{1}{2}\int_0^{tan^{-1}[\frac{f(x)}{x}]} \rho^{\,2}(\theta)\,d\theta$$

of the Theorem. Hence the antiderivative will be obtained as follows,

$$F(x) = \frac{x\,f(x)}{2} - \frac{1}{2}\int_0^{tan^{-1}[\frac{f(x)}{x}]} tan\,\theta\ \ d\theta\ = \quad \frac{x\,f(x)}{2} + \frac{1}{2}\ \ ln(\ cos\theta\ )$$
$$\Bigg|_{0}^{tan^{-1}[\frac{f(x)}{x}]} = \frac{x\,f(x)}{2} + \frac{1}{2}\ ln\ (\frac{x}{\sqrt{x^2+y^2}})$$

and therefore

$$F(x) = \frac{x\,f(x)}{2} + \frac{1}{2}\ ln\ (\frac{x}{\sqrt{x^2+y^2}})$$

is the required antiderivative for of the function $y = f(x)$ implicitly defined by relation (7).

<u>Algebraic Verification of the Solution:</u> To show that the derivative of $F(x)$ as expressed above is $y = f(x)$, as an easy and routine exercise to the reader to use rules of differentiation to show $\frac{d}{dx}\ [ln\ (\frac{x}{x^2+y^2})] = \frac{y^2 - xyy\prime}{x\,(x^2+y^2)}$. Having shown this, if we replace the denominator $x\,(x^2+y^2)$ by y from elation (5), we get $\frac{d}{dx}\ [ln\ (\frac{x}{x^2+y^2})] = \frac{y^2-xyy\prime}{y} = \frac{y+xy\prime}{2}$. Therefore, $F\,'(x) = \frac{d}{dx}[\frac{xy}{2} + \frac{1}{2}\ \ ln\ (\frac{x}{x^2+y^2})] = \frac{y+xy\prime}{2} + \frac{y+xy\prime}{2} = y$, or $F\,'(x) = y = f(x)$, meaning $F(x)$ is indeed an antiderivative of $f(x)$, and the verification is complete.

<u>Application 1</u> The method can provide a more clearcut way to do the integration $\int\sqrt{a^2 - x^2}\ , 0 \le |x| < a$.
The reader may recall that this this integration is usually accomplished by the method of substitution of $x = a\ cos\ \theta$ or
$x = a\ sin\ \theta$. However, if we apply the relation Theorem of the article to the cartesian relation $y = \sqrt{a^2 - x^2}$, representing the upper half circle of radius a , the polar version of the relation is simply $\rho = a$, and therefore the Theorem implies that an antiderivative for $y = f(x)$ is,

$$F(x) = \frac{x\,y}{2} - \frac{1}{2}\int_0^{tan^{-1}(\frac{y}{x})} a^2\ \mathrm{d}\theta\ =$$
$$= \frac{x\,y}{2} - \frac{1}{2}\ a^2\theta = \frac{x\,y}{2} - \frac{a^2}{2}\,tan^{-1}\left(\frac{y}{x}\right)$$
$$\frac{x\,y}{2} - \frac{1}{2}\ a^2\theta = \frac{x\,y}{2} - \frac{a^2}{2}\,tan^{-1}\left(\frac{\sqrt{a^2-x^2}}{x}\right)$$
$$= \frac{x\,\sqrt{a^2-x^2}}{2} - \frac{1}{2}\ sin^{-1}(\frac{x}{a})$$

Therefore, the general antiderivative for $y = \sqrt{a^2 - x^2}$ would be

$$F(x) = \frac{x\sqrt{a^2-x^2}}{2} - \frac{1}{2}\, sin^{-1}(\frac{x}{a}) + C.$$

Application 2 The next application shows how the method under discussion can be used to find a trigonometric approximation can be obtained for the function $y = ln(x)$.

We first conclude the following corollary from the Theorem:

Corollary: For any positive number x, we have

$$Ln\, x \;=\; -\int_{\frac{\pi}{4}}^{tan^{-1}(\frac{1}{x^2})} csc\,(2\theta)\, \mathrm{d}\theta\,. \qquad (*)$$

Proof: Consider the function $f(x) = \frac{1}{x}$ and let $a = 1$. Then upon substitutions $x = \rho\, cos\,\theta$ and $y = \rho\, sin\,\theta$ in the equation $y = \frac{1}{x}$ we get $\rho\, sin\,\theta = \frac{1}{\rho\, cos\,\theta}$, therefore $\rho^{\,2} = \frac{1}{sin\,\theta\, cos\,\theta} = 2\, csc\,(2\,\theta)$.

Now, setting $a = 1$ and considering that for any positive number x we have $x\, f(x) = 1$ the formula (1) of the Theorem implies

$$F(x) = \frac{x\, f(x)}{2} - \frac{1}{2}\int_{tan^{-1}\left(\frac{f(1)}{1}\right)}^{tan^{-1}\left(\frac{f(x)}{x}\right)} \rho\,(\theta)^2 d\theta =$$

$$\frac{1}{2} - \frac{1}{2}\int_{\frac{\pi}{4}}^{tan^{-1}\left(\frac{f(x)}{x}\right)} 2\csc(2\theta)\, d\theta =$$

$$\frac{1}{2} - \int_{\frac{\pi}{4}}^{tan^{-1}\left(\frac{1}{x^2}\right)} \csc(2\theta)\, d\theta$$

To show that above proposed antiderivative is indeed an antiderivative of $y = \frac{1}{x}$ we will show that the condition

$$\frac{d}{dx}[\int_{tan^{-1}\left(\frac{f(a)}{a}\right)}^{tan^{-1}\left(\frac{f(x)}{x}\right)} \rho\,(\theta)^2 d\theta\,] = xy' - y. \qquad (2)$$

of the Theorem holds. To this end considering that $y' - \frac{1}{x^{\,2}}$, the right side of (2) is simplified as $x(-\frac{1}{x^{\,2}}) - \frac{1}{x} =) - \frac{2}{x}$, and the left side amounts to

$$\frac{d}{dx}\left[\int_{\frac{\pi}{4}}^{tan^{-1}\left(\frac{1}{x^2}\right)} 2\csc\;(2\theta)\, d\theta\,\right] = \; 2\,\frac{d}{dx}\,[\tfrac{1}{2} Ln\,(\tan(\theta)]\left|\begin{matrix} tan^{-1}\left(\frac{1}{x^{\,2}}\right) \\ \frac{\pi}{4} \end{matrix}\right.\Bigg] =$$

$$\frac{d}{dx}[\,Ln\frac{1}{x^{\,2}}] = -2\,\frac{d}{dx} Ln\;\,x\,] = -\frac{2}{x}.$$

Hence, the condition (2) holds, and indeed

$$F(x) = \frac{1}{2} - \int_{\frac{\pi}{4}}^{tan^{-1}\left(\frac{1}{x^{\,2}}\right)} \csc(2\theta)\, d\theta$$

Is an antiderivative of $f(x) = \frac{1}{x}$. Next, since both functions $Ln\, x$ and $F(x) - \frac{1}{2} = -\int_{\frac{\pi}{4}}^{tan^{-1}\left(\frac{1}{x^{\,2}}\right)} \csc(2\theta)\, d\theta$ are particular antiderivatives of the $f(x)=\frac{1}{x}$ with initial condition $y(1)=0$, they should be equal functions, that is, $Ln\,x = F(x)-\frac{1}{2} = -\int_{\frac{\pi}{4}}^{tan^{-1}\left(\frac{1}{x^2}\right)} \csc(2\theta)\,d\theta$, and the proof of the Corollary is complete.

<u>The Trigonometric Approximation of $Ln\ (x)$</u>

Now for any $x>0$, to obtain the claimed approximation for $Ln\,x$ it is only enough to express the definite integral (*) in the Corollary as the limit of one of the known Riemann sums over the interval $[\frac{\pi}{4}-\theta_x]$, say the limit of the right Riemann sum], to get

$$Ln\ (x) \ = (\frac{\pi}{4}-\theta_x) \lim_{n\to\infty} \sum_{i=1}^{n} \csc 2[\frac{\pi}{4}+\frac{\boldsymbol{i}}{\boldsymbol{n}}(\theta_x-\frac{\pi}{4})] \ .$$

Using the identity $\csc\left(\frac{\pi}{2}-\alpha\right) = \sec\alpha$ the above approximation will look easier, as our claimed approximationas:

$$Ln\ (x) = \left(\frac{\pi}{4}-\theta_x\right) \lim_{n\to\infty} \sum_{i=1}^{n} \sec[\frac{\boldsymbol{2i}}{\boldsymbol{n}}(\frac{\pi}{4}-\theta_x)] \ . \qquad \textbf{(**)}$$

We will also get two more similar approximation for $Ln\ (x)$, by expressing the same definite integral as the limits of left and mid-point Riemann sums.