Finding Equations of Tangent Lines to Polynomial Curves Avoiding Calculus

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In this article, we first present a Lemma, asserting how to find equations of tangent lines to any parabola represented by a function $f(x) = a\, x^2 + bx + c$, without any reference to derivative of $f(x)$. Since in BC Pre-Calculus 11 students deal with parabolas in their course, the Lemma will enable them to find equation of the tangent lines to any parabola at any of their points $(u, f(u))$ .Then we will present a Proposition enabling BC Pre-Calculus 12 students (or students of any level who know the method of long division of polynomials) to find equation of the tangent line to any polynomial curve represented by a function of the form

$$f(x) = a_1\, x^n + a_2 x^{n-1} + \cdots + a_n x + a_{n+1}$$

at any point $(u, f(u))$ of $f(x)$. Again, the assertion of the Proposition doesn't make any reference to derivative of f(x).

For convenience, in the rest of this article I will adopt the notation T_u to represent the tangent line to the graph of any function f(x) at the point $(u, f(u))$ on the graph.

Finding Equations of Tangent Lines for Parabolas at Pre-Calculus 11 Level

Lemma 1: Let $f(x) = a\, x^2 + bx + c$ represent a parabola and let $P(u, f(u))$ be an arbitrary point on the parabola. For any reasonably small number Let δ (say $\delta = 1$ if you wish), consider the two points $A(u - \delta\, , f(u - \delta\,)\,)$ and $B(u + \delta\, , f(u + \delta)\,)$ on the parabola which are to the left and the right of $P(u, f(u))$ respectively. Then the secant line segment AB on the parabola is parallel to the tangent line T_u at $\big(u, f(u)\big)$. Hence the slope of the secant segment AB , which we know is given by $m = \frac{f(u+\delta\,) - f(u-\delta\,)}{2\delta}$, will be the same as the slope of the tangent line tangent line T_u at (P) **.**
Therefore the tangent line can be expressed as $y - f(u) = m\,(x - u)$,
This equation is simplified into

$$y = \frac{f(u+\delta) - f(u-\delta)}{2\delta}(x - u) + f(u), \qquad (*)$$

presenting equation of the tangent line T_u is at the point $P(u, f(u))$.

Before I bring the proof of the Lemma, let us first apply it to a specific example.

Example 1 To find the equation of the tangent line at the point $P(2,5)$ on the graph of the parabola represented by $f(x) = 2\,x^2 - 3x + 3$, chose δ to be the most convenient positive number, that is $\delta = 1$, and consider the points A (2-1, f (2-1)) =A (1,2) and B (2+1, f (2+1) = B (3,12) on the graph of $f(x)$. Then by formula (1) of Lemma 1, the tangent line at the point $P(2,5)$ to the parabola is $y = \frac{f(u+\delta)-f(u-\delta)}{2\delta}\,(x-2) + f(u)$. Or $y = \frac{12-2}{2(1)}\,(x-2) + 5$, which is simplified into $y = 5x - 5$.

As seen above neither the statement of the Lemma, and nor its application make a mention of the even don't the word "derivative", However, as we will see bellow, the proof of the Lemma does use the most elementary fact about the derivatives, that slope of the tangent line T_u at $(u, f(u))$ is $f'(u)$.

<u>Proof of the Lemma</u> Let $f(x) = a\,x^2 + bx + c$ be the function representing our parabola, and T_u be the tangent line at a chosen point $P(u, f(u))$ on the parabola. To conclude the assertion of the Lemma, let δ an arbitrary positive number. We first observe that the slope of the secant segment AB on the parabola, where $A(u-\ \delta, f(u-\ \delta))$ and B$(u+\ \delta, f(u+\ \delta))$ is given by $m = \frac{f(u+\delta)-f(u-\delta))}{(u+\delta)-(u-\delta)}$. Or by

$$m = \frac{a(u+\ \delta)^2 + b(u+\ \delta)) + c - a(u-\ \delta)^2 - b(u-\ \delta) - c}{2\delta}$$

$$= \frac{au^2 + 2a\delta u + a\,\delta^2 + bu + b\,\delta\ + c - au^2 + 2a\delta u - a\,\delta^2 - bu + b\,\varepsilon\delta - c}{2\delta}$$

$$= \frac{4a\varepsilon u + 2b\,\delta}{2\delta} = 2au + b.$$

On the other hand, we know the most elementary fact in Calculus that slope of the tangent line at the point $(u, f(u))$ is $f'(u)$. Since the derivative of $f(x) = a\,x^2 + bx + c$ is given by $f('x) = 2ax + b$, the slope of the tangent line at $P(u, f(u))$ is $f('x) = 2au + b$. Since both slopes of the secant line AB and the tangent line T_u are the same the proof of the Lemma is complete.

<u>*Finding Equations of Tangent Lines to all Polynomial Curves avoiding Calculus*</u>

The following Proposition is a much tool than Lemma 1, as it covers the particular case of parabolas (a graphs of quadratic functions) case of as a particular case, when $n = 2$.

Proposition Let $f(x) = a_1 \, x^n + a_2 \, x^{n-1} + \cdots + a_{n-1}x + a_n$, $n \geq 2$ be any polynomial function. Let $P(u, f(u))$ be any point on the graph of the polynomial function and let T_u represent the tangent line to the graph of $f(x)$ at $P(u, f(u))$. To find equation of T_u , use the Method of Long Division for polynomials and divide the given polynomial $f(x)$ by the trinomial $(x-u)^2 = x^2 - 2ux + u^2$ and write the division statement as

$$f(x) = q(x)(x-u)^2 + mx + b. \qquad (**)$$

Then $y = mx + b$ is the required equation of the tangent line T_u .

Again, let us first apply the proposition to an example.

Example 2 To find equation of the tangent line T_2 at the point $P(2,17)$ on the graph of the polynomial function defined by $f(x) = x^2 + 4x + 12$, we first use the method of long division and divide the polynomial $x^4 + 2x - 3$ by

$$(x-2)^2 = x^2 - 4x + 4$$

Then $q(x) = x^3 + 2x - 3$ and $R(x) = 34x - 51$ Will be the quotient and the remainder of the long division respectively. Hence the division statement can be expressed as

$$x^4 + 2x - 3 = (x^2 + 4x + 12)\,(x-2)^2 + 34x - 51.$$

Therefore, by above Proposition equation of the tangent line T_2 to the polynomial curve represented by $f(x)$ at $P(2,17)$ is y= $34x - 51$.

Proof of the Proposition First from equation (**) in the Proposition we observe that $f(u) = mu + b$, and therefore $b = f(u) - mu$.
Next, differentiating both sides of (2), using elementary rules of differentiation, and we get

$$f'(x) = q'(x)(x-u)^2 + 2q(x)(x-u) + m,$$

from which we conclude $f'(u) = m$ is the slope of the tangent line T_2 .
Therefore $y - f(u) = m(x-u)$ will be equation of the tangent line T_2 .
Now, since $b = f(u) - mu$, the latter equation is simplified into
$y = mx + f(u) - mu) = mx + b$, and the proof of the Proposition is complete.

Remark A good question regarding earlier Lemma 1 would be whether property (*) proved earlier for quadratic functions in Lemma 1 could be a characterization for parabolas? Otherwise said, whether we can prove the following,

Conjecture: If the property

$$f'(x) = \frac{f(x+\delta) - f(x-\delta)}{2\delta} \qquad (*)$$

holds for all real numbers x and all positive numbers δ, then $y = f(x)$ is necessarily a quadratic function.

To support this conjecture, first of all, it is easy to show that the property (*) is invariant under horizontal and vertical shifts, as well as horizontal and vertical expansions for $y = f(x)$. Using this fact one can begin to support the conjecture by proving the following,

Lemma 2 If (*) holds for all x and for every positive δ, then an entirely differentiable function $y = f(x)$, then graph of $f(x)$ can't have any inflection points. Another words, the graph of $f(x)$ is either entirely concave up, or concave down.

Proof: Assume otherwise, that is let $(a, f(a))$ be an inflection point for $f(x)$. Since we are assuming that the property $f'(x) = \frac{f(x+\delta)-f(x-\delta)}{2\delta}$ (*)
Holds for all real numbers x and all positive numbers δ , and since (as we mentioned before) this property is invariant under both horizontal and vertical transformation of $f(x)$, we can assume without loss of generality that the inflection point is at the origin, that is $(a, f(a)) = (0,0)$. Also since the property is invariant under vertical multiplication of $f(x)$ by -1, we can assume that for sufficiently small positive number $\delta > 0$, the graph of $f(x)$ is concave up over [0, δ] and concave down over $[-\delta, 0]$, in which case $f(x)$ will be an increasing function over the interval $[-\delta, \delta]$. Observe that here $f'(x) = 0$ will not be possible, otherwise (*) will imply $f(-\delta) = f(\delta)$, which is impossible under the present circumstance. Therefore $f'(0) = \frac{f(\delta)-f(-\delta)}{2\delta} > 0$, from which we conclude $f(\delta) > f(-\delta)$. On the other hand, let $L_0(x) = f'(0)x$ be the equation of the tangent line at the inflection point (0,0) to the graph of $f(x)$. Then, $f(x)$ being increasing and concave up over [0, δ] we Must have will have $L_0(\delta) = f'(0)\delta < f(\delta)$. Similarly, since $f(x)$ is increasing and concave down over $[-\delta, 0]$, we must have $L_0(-\delta) = -f'(0)\delta > f(-\delta)$. The two latter inequalities together imply that

$$f(\delta) - f(-\delta) > f'(0)\delta + -f'(0)\delta = 2f'(0)\delta.$$

From this we conclude that $f'(0) > \frac{f(\delta)-f(-\delta)}{2\delta}$ which contradicts $f(x)$ having property (*). Hence the proof of the lemma is complete.

Of course, there will be still a long way to go to prove that property (*) is a characterization for quadratic functions if we only assume that $f(x)$ is an entirely differentiable function. However, I have concluded the conjecture is valid if we assume $f(x)$ is a polynomial. That is,

Theorem: If $f(x) = a_1\, x^n + a_2\, x^{n-1} + \cdots + a_{n-1}x + a_n$ is a polynomial of degree $n > 2$, then there exists a real number a such that the property

$$f'(a) = \frac{f(a+\delta)-f(a-\delta)}{2\delta} \qquad (*)$$

will not hold for all positive numbers δ.

Proof: First of all, $f(x)$ can't be a polynomial of odd degree, because then $f''(x)$ will also be of an odd degree, and the equation $f''(x) = 0$ will have at least one real root $x = a$, for which $f''(a) = 0$; implying $(a, f(a))$ is an inflection point for $f(x)$

which contradicts the assertion of Lemma 2. Therefor $f(x)$ must be of even order, that is of the form $f(x) = a_2\ x^{2k} + a_3\ x^{2k-1} + \cdots + a_n$. Next, since $f'(x) = 2ka_2x^{2k-1} + (2k-1)a_3x^{2k} + \cdots + a_{n-1}$ is a polynomial of odd degree, the equation $f'(x) = 0$ has at least one root, meaning there is at least one extremum point $(a, f(a))$ for $f(x)$. Indeed, this extremum point will be a unique, because any function with more than one extremum point is bound to have a reflection point, again contradicting Lemma 2. Now, since we already know that the property (*) is invariant under vertical multiplication by -1, we can assume, without loss of generality, that the extremum point $(a, f(a))$ is a minimum point, implying that the graph of $f(x)$ is simply a U-shape graph (except that form both left and right gradually go to infinity horizontally). Moreover, since we know the property (*) is also invariant under both horizontal and vertical transformation of $f(x)$, again without loss of generality we can assume that the unique minimum point $f(x)$ is at the origin, that is $(a, f(a)) = (0,0)$. Therefore, if $f(x)$ has (*) property then (for all possible $\delta > 0$ we should have $f'(0) = \frac{f(\delta)-f(-\delta)}{2\delta} = 0$. Therefore, for all $\delta > 0$ we should have $f(\delta) = f(-\delta)$, and this can only happen if $f(x)$ is defined in terms of even powers of x. That is, it is of the form $f(x) = a_2\ x^{2k} + a_4\ x^{2k-2} + \cdots + a_{2k}x^2$. Note that there will be no constant term, because $f(0) = 0$. It is now easy to see that if for arbitrary $\delta > 0$ we set $a = \delta$ then the property (*), that is the equation $f'(\delta) = \frac{f(\delta+\delta)-f(\delta-\delta)}{2\delta} = \frac{f(2\delta)-f(0)}{2\delta} = \frac{f(2\delta)}{2\delta}$ which can be expressed as $2\delta f'(\delta) = f(2\delta)$ will lead to a contradiction. Because as polynomials in variable δ, the left polynomial $2\delta f'(\delta)$ will have a leading coefficient of $2^{2k}ka_2$, whereas the leading coefficient of $f(2\delta)$ will be $2^{2k}a_2$, and this can only happen if $k = 1$, meaning $f(x)$ is a quadratic function. This completes the proof of the Theorem.

To further support the conjecture, it is worthwhile to note that neither of the most popular entirely differentiable transcendental function we know enjoy property (*).

For example, no trigonometric function can have property (*), because they all have inflection points contrary to the assertion of the Lemma 2.

Also for the exponential functions $\mathrm{f}(x) = e^{\alpha x}$, with $\alpha > 0$, to show that property doesn't (*) doesn't hold, assume otherwise, then for $a = 0$ any positive δ, we should have $f'(0) = \alpha = \frac{e^{\delta}-e^{-\delta}}{2\delta}$, or $2\alpha\delta = e^{\delta} - e^{-\delta}$. Now, differentiating both sides of the latter relation with respect to variable δ we get $2\alpha = e^{\delta} + e^{-\delta}$. However, if we now choose for $\delta = ln\ \alpha$, we get $2\alpha = 2\alpha\ + \frac{1}{\alpha}$, or $0 = \frac{1}{\alpha}$ which is absurd. Hence no exponential function $\mathrm{f}(x) = e^{\alpha x}$, $\alpha > 0$ can't have property (*). Similar argument shows $\mathrm{f}(x) = e^{\alpha x}$, $\alpha < 0$ can't have property (*) either.

I close the article by bringing an exercise, showing property (*) can't hold for four more familiar functions representing familiar curves. The first three functions have rather more restricted domains and perhaps the more interesting case would be th last one for the hyperbolic function $f(x) = Cosh\ x - 1$, as its graph pretty much resembles parabolas represented by $g(x) = ax^2, a > 0$.

Exercise (**a)** Show that for the symmetric half unit circle curve defined by $f(x) = \sqrt{1-x^2} + 1,\ -1 \le x \le 1$, the property $f'(a) = \frac{f(a+\delta)-f(a-\delta)}{2\delta}$ doesn't hold for $a = \frac{1}{2}$ and any $0 < \delta < \frac{1}{2}$.

[Hint: Perhaps the easiest way would be by rationalizing the numerator of the fraction on the right side of the equality $f'\left(\frac{1}{2}\right) = \frac{f\left(\frac{1}{2}+\delta\right)-f\left(\frac{1}{2}-\delta\right)}{2\delta}$]

(b) Show that the square root function $f(x) = \sqrt{x}$ can't satisfy the (*) property for $a = 1$ and any $0 < \delta < \frac{1}{2}$.

(c) Show that the assumption that property (*) holds for the function $f(x) = \ln x$, for $a = 1$ and $\delta = \frac{1}{2}$ leads to the contradiction that the base $e = 3$.

(d) Show that the function $f(x) = Cosh(x) - 1 = \frac{e^x - e^{-x}}{2}$-1 whose graph resembles that of the parabola defined by $g(x) = x^2$, doesn't have the property (*).

[Hint: Show that if property (*) holds for $a = \delta = ln\ 2$, this time we get $e = 4$].

Exciting Update

Sometime, around early October (2025) I came up with a solid proof for my conjecture, stated on page 3, but for a good reason I will not bring it up here for a long while!