A Method of Implicit Integration

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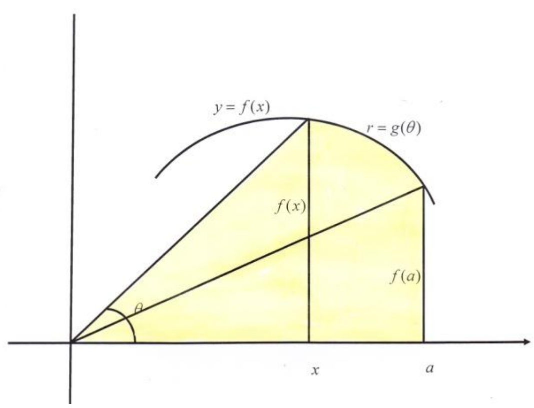
As far as my decades of teaching Calculus is concerned, most likely the following is the deepest attempt to integrate a function defined implicitly, even though (just like in the case of implicit differentiation) here too the anti-derivatives are obtained in terms of both x and y in general. So here first bring the first (definite integral) version of the method,

Theorem 1 [A. Astaneh] Let $y = f(x)$ be a differentiable function defined over the open interval $(0,\infty)$ and let $r = g(\theta)$ be the polar representation of a curve $y = f(x)$ upon substitutions $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Note that here, if the function $y = f(x)$ is defined by an implicit relation, say $R(x,y) = 0$, it is then understood that $r = g(\theta)$ is the polar version of this implicit relation. Then, for any $a > 0$ the following will give an anti-derivative for $f(x)$, which in general (and eventually) will be expressed in terms of both x and y.

$$F(x) = \frac{xf(x)}{2} - \frac{1}{2}\int_{tan^{-1}[\frac{f(a)}{a}]}^{tan^{-1}[\frac{f(x)}{x}]} r(\theta)^2 d\theta.$$



Proof: In the above diagram, let $F_1(x)$ be the anti-derivative for $y = f(x)$ defining the area under the graph of $f(x)$ over the interval $[x, a]$, and let A_x denote this area. Also, let A_θ denote the polar area enclosed by the origin, the polar curve $r = g(\theta)$, and the polar rays of line defined by equations

$\theta_1 = \tan^{-1}(f(a)/a)$, $\theta_2 = \tan^{-1}(f(x)/x)$. Then the shaded area in the above diagram can be represented by the flowing equal expressions,

$$\frac{xf(x)}{2} + A_x = \frac{a\ f(a)}{2} + A_\theta . \qquad (*)$$

Using the anti-derivative $F_1(x)$, for $f(x)$, and applying the Fundamental Theorem of Calculus upon it to find the area under the curve of $y = f(x)$ over the interval $[x, a]$, and also using the popular formula

$$\frac{1}{2}\int_{\theta_1}^{\theta_2} r(\theta)^2\ d\theta$$

to express the , the above equation (*) can be expressed as

$$\frac{x\ f(x)}{2} + F_1(a) - F_1(x) = \frac{a\ f(a)}{2} + \frac{1}{2}\int_{\tan^{-1}[f(a)/a]}^{\tan^{-1}[f(x)/x]} r(\theta)^2\ d\theta .$$

Since by assumption $F_1(a) = 0$, it follows,

$$F_1(x) = \frac{x\ f(x)}{2} - \frac{a\ f(a)}{2} - \frac{1}{2}\int_{\tan^{-1}[f(a)/a]}^{\tan^{-1}[f(x)/x]} r(\theta)^2\ d\theta ,$$

Therefore

$$F(x) = F_1(x) + \frac{af(a)}{2} = \frac{xf(x)}{2} - \frac{1}{2}\int_{tan^{-1}[\frac{f(a)}{a}]}^{tan^{-1}[\frac{f(x)}{x}]} r(\theta)^2 d\theta.$$

is an anti-derivative for $y = f(x)$, and the proof is complete.

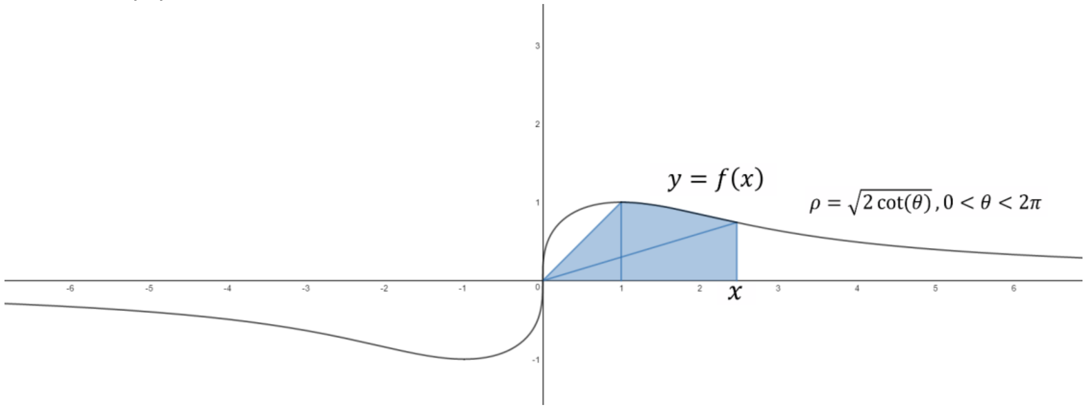
Given a specific implicit relation, and when the real number a is pre-selected, than the above Theorem works like it does in the following four examples of different nature.

<u>Example 1:</u> Let $y = f(x)$ be the function implicitly defined by the relation

$$y(x^2 + y^2) = 2x\ ,\ \ x \geq 0, y \geq 0 . \qquad (1)$$

Find an anti-derivative $F(x)$ for the function $y = f(x)$ with initial condition $F(1) = 0$.

<u>Analytic Solution:</u> Upon polar substitutions of $x = \rho\cos(\theta)$ and $y = \rho\sin(\theta)$ in (1), the relation is converted to the polar form $\rho^2 = 2\cot(\theta)$, or $\rho = \sqrt{2\cot(\theta)}$ withthe following polar graph, which at the same time represents graph of relation (1),



Now let $F(x)$ be the anti-derivative for $y = f(x)$ defining the area under its graph over the interval $[1, x]$, and let A_x denote that area. Also, let A_θ denote the polar area between the polar curve, the origin, and between arrays of lines $\theta = \pi/4$ and $\theta = \tan^{-1}[f(x)/x]$. Then the above figure shows the validity of the following relation between the two areas A_x and A_θ,

$$\frac{1f(1)}{2} + A_x = \frac{x\,f(x)}{2} + A_\theta .$$

By the Fundamental Theorem of Calculus, and the popular polar formula for the polar area A_θ, the above relation implies,

$$\frac{1}{2} + F(x) - F(1) = \frac{x\,f(x)}{2} + \frac{1}{2}\int_{\tan^{-1}[f(x)/x]}^{\pi/4} \rho(\theta)^2\, d\theta ,$$

$$F(x) = \frac{x\,f(x)}{2} - \frac{1}{2} + \frac{1}{2}\int_{\tan^{-1}[f(x)/x]}^{\pi/4} 2\cot(\theta) d\theta ,$$

$$F(x) = \frac{x\,f(x)}{2} - \frac{1}{2} + \ln(\sin\theta)\Big|_{\tan^{-1}[f(x)/x]}^{\pi/4} ,$$

$$F(x) = \frac{x\,f(x)}{2} - \frac{1}{2} - \frac{1}{2}\ln(2) - \ln\frac{y}{\sqrt{x^2 + y^2}}$$

$$F(x) = \frac{x\,f(x)}{2} + \ln(\frac{\sqrt{(x^2 + f(x)^2)}}{\sqrt{2}\, f(x)}) - \frac{1}{2}$$

Therefore the solution is $F(x)=\frac{x\,y}{2}+\ln(\frac{\sqrt{(x^2+y^2)}}{\sqrt{2}\,y})-\frac{1}{2}$, where $y=f(x)$ is the function implicitly defined in (1).

Algebraic Verification of the Solution: It is enough to show that the derivative of the right hand side of the equation $F(x)=\frac{x\,y}{2}+\ln(\frac{\sqrt{(x^2+y^2)}}{\sqrt{2}\,y})-\frac{1}{2}$ is $y=f(x)$.

To this end, in the following, when trying to simplify the derivative of the right hand side of the equation we will make use the original implicit relation (1). That is, when the expression $y(x^2+y^2)$ shows up as the denominator of a fraction we will replace it by $2x$.

$$\frac{d}{dx}\left[\frac{xy}{2}+ln\left(\frac{\sqrt{x^2+y^2}}{\sqrt{2}\ y}\right)-\frac{1}{2}\right]=\frac{d}{dx}\left[\frac{xy}{2}+\frac{1}{2}\ ln(x^2+y^2)-ln\,y\ -\ln 2-\frac{1}{2}\right]$$

$$=\frac{y+xy'}{2}+\frac{2x+2yy'}{2(x^2+y^2)}-\frac{y'}{y}$$

$$=\frac{y+xy'}{2}+\frac{y\left(x+yy'\right)-y'(x^2+y^2)}{y(x^2+y^2)}$$

$$=\frac{y+xy'}{2}+\frac{y\left(x+yy'\right)-y'(x^2+y^2)}{2x}$$

$$\frac{xy+x^2y+xy+y^2y'-x^2y-y^2y'}{2x}=y$$

Example 2: Let $y=f(x)$ be the function implicitly defined by the relation

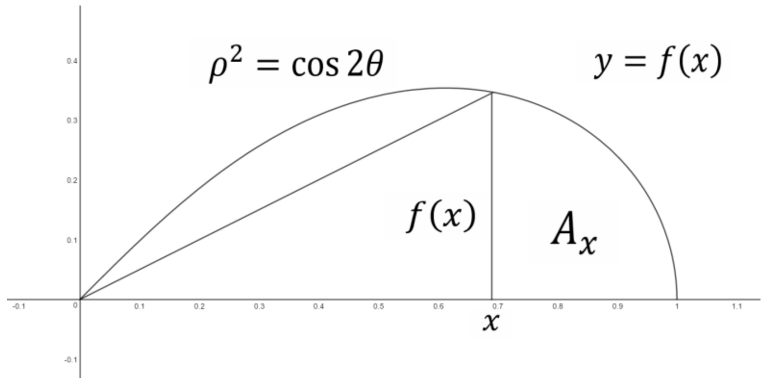
$$(x^2+y^2)^2=x^2-y^2\ ,\ \ 0\le x\le 1,\ y\ge 0. \quad (2)$$

Find an anti-derivative for $y=f(x)$ satisfying $F(1)=0$.

Analytical solution: Substituting $x=\rho\cos\theta$ and $x=\rho\sin\theta$ in (2), the Cartesian relation is converted into a polar relation as follows

$$(x^2+y^2)^2=x^2-y^2\rightarrow\rho^4=\rho^2\cos^2\theta-\rho^2\sin^2\theta=\rho^2\cos 2\theta\rightarrow\ \rho^2=\cos 2\theta.$$

The conditions $0\le x\le 1$ and $y\ge 0$ mean, here we are concerned with the upper right part of the Lemniscate curve $\rho^2=\cos 2\theta$ as seen below,



Let $F(x) = \int_x^1 f(x)dx$, $F(1) = 0$, be the desired anti-derivative for $y = f(x)$ defining the area under the graph over the interval $[x, 1]$, for $x > 0$, and let A_x denote that area under the curve $y = f(x)$ over the interval $[x, 1]$. Also, let

$$A_\theta = \frac{1}{2}\int_0^{\tan^{-1}[f(x)/x]} \rho(\theta)^2 \, d(\theta)$$

be the polar area enclosed by the Lemniscate curve, the x-axis, and the lines represented by the polar rays $\theta = 0$ and $\theta = \tan^{-1}[f(x)/x]$. Then, as seen in the diagram, the following relation holds between the two areas A_x and A_θ,

$$A_x = A_\theta - \frac{x\, f(x)}{2}.$$

By the Fundamental Theorem of Calculus the left hand side is just $A_x = F(1) - F(x)$. Therefore, also using the polar definite integral formula for the area A_θ, the above relation can be expressed as

$$F(1) - F(x) = \frac{1}{2}\int_0^{\tan^{-1}[f(x)/x]} \cos(2\theta)\, d\theta - \frac{x\, f(x)}{2}.$$

Hence, considering $F(1) = 0$, we have

$$F(x) = \frac{x\, f(x)}{2} - \frac{1}{4}[\sin(2\theta)]\Big|_0^{\tan^{-1}[f(x)/x]}.$$

Or,

$$F(x) = \frac{xf(x)}{2} - \frac{xf(x)}{2[x^2 + f(x)^2]}.$$

Therefore the solution is $F(x) = \dfrac{x\,y}{2} - \dfrac{xy}{2(x^2 + y^2)}$, where $y = f(x)$ is the function implicitly defined in (2).

Algebraic Verification of the Solution : Again, it is enough to show that the derivative of the right hand side of $F(x) = \frac{x\,y}{2} - \frac{xy}{2(x^2+y^2)}$ is $y = f(x)$. To this end, this time in our verification when the expression $(x^2+y^2)^2$ shows up as a denominator of a fraction we will make use the assumed original implicit relation (2) and replace it by (x^2-y^2) .

$$\frac{d}{dx}[\frac{xy}{2} - \frac{xy}{2(x^2+y^2)}] = \frac{1}{2}(y+xy') - \frac{1}{2}\frac{(y+xy')(x^2+y^2) - xy(2x+2yy')}{(x^2+y^2)^2} =$$

$$\frac{1}{2}(y+xy') - \frac{1}{2}\frac{xy'(x^2-y^2) - y(x^2-y^2)}{(x^2-y^2)} = \qquad .$$

$$\frac{1}{2}(y+xy') - \frac{1}{2}(xy'-y) = y.$$

Remark: Incidentally the implicit relation $(x^2+y^2)^2 = x^2 - y^2$ in above Example 2 can be solved for $y \geq 0$, but in a not so handsome expression as follows,

$$y = \frac{1}{\sqrt{2}}\sqrt{\sqrt{8x^2+1} - (2x^2+1)}\,.$$

Therefore, the above solution to Example (2) implies we have been able to <u>explicitly</u> integrate the rather complicated looking integral,

$$\int \frac{1}{\sqrt{2}}\sqrt{\sqrt{8x^2+1} - (2x^2+1)}\,dx, \quad 0 < x \leq 1,$$

as Example 2 implies that the solution is,

$$\int \frac{1}{\sqrt{2}}\sqrt{\sqrt{8x^2+1} - (2x^2+1)}\,dx = \frac{xf(x)}{2} - \frac{xf(x)}{2[x^2+f(x)^2]} + C\,,$$

Where $f(x) = \frac{1}{\sqrt{2}}\sqrt{\sqrt{8x^2+1} - (2x^2+1)}$ happens to be the integrand itself. Upon substitution of this expression for $f(x)$, and algebraic simplification, the above integral is obtained as,

$$\int \sqrt{\sqrt{8x^2+1} - (2x^2+1)}\,dx = \frac{x\sqrt{\sqrt{8x^2+1} - (2x^2+1)}(\sqrt{8x^2+1} - 3)}{2(\sqrt{8x^2+1} - 1)} + C.$$

The author, for one, doesn’t recall any previously known integration technique to handle the above explicit integral. Consider this as a side job bonus from the method of implicit integration presented in this article.

Example 3 Let $y = f(x)$ be the function implicitly defined by the relation

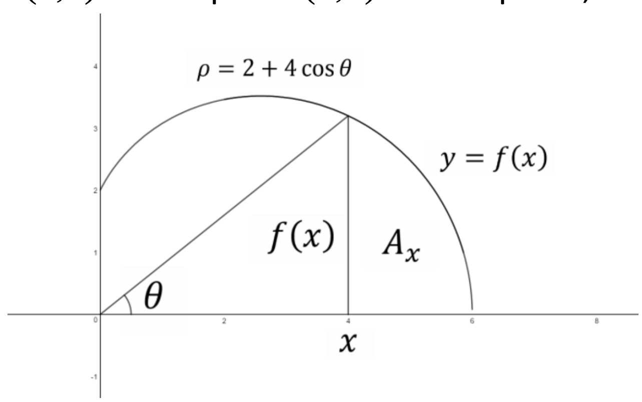
$$(x^2 + y^2 - 4x)^2 = 4(x^2 + y^2) \ , \ 0 < x \leq 6 , \ y \geq 0 . \quad (3)$$

Find the anti-derivative for $y = f(x)$ satisfying $F(6) = 0$.

Analytical solution: Again, substituting $x = \rho\cos\theta$ and $x = \rho\sin\theta$ in (3), the Cartesian relation is converted into a polar relation as follows,

$$(\rho^2 - 4\rho\cos\theta)^2 = 4\rho^2 \quad \to \quad (\rho - 4\cos\theta)^2 = 4 \quad \to \rho = 4\cos\theta \pm 2 .$$

Since the case $\rho = 2 - 4\cos\theta$ implies $x = \rho\cos\theta = \cos(\theta)\,(2 - 4\cos(\theta))$ from which the value $x = 6$ will never be obtained for x, the requirement $F(6) = 0$ implies that here we are concerned with the polar equation $\rho = 2 + 4\cos\theta$ whose graph is part of the Limacon in the first quadrant curving counter-clockwise from point $(6,0)$ to the point $(0,2)$ on the plane, as seen below



Again, let $F(x)$ be the anti-derivative for $y = f(x)$ defining the area under its graph over the interval $[x, 6]$, and let A_x denote that area. Also, let A_θ denote the polar area enclosed by the Lemicon, the x-axis, and the line with the angle of inclination $\theta = \tan^{-1}[f(x)/x]$. Then the above figure shows the validity of the following relation between the two areas A_x and A_θ ,

$$A_x = A_\theta - \frac{x\,f(x)}{2} .$$

By the Fundamental Theorem of Calculus, and the polar coordinate formula for A_θ, the above relation can be expressed as

$$F(6) - F(x) = \frac{1}{2}\int_0^{\tan^{-1}[f(x)/x]} [2 + 4\cos(\theta)]^2 \, d\theta \ - \frac{x\,f(x)}{2} .$$

Therefore, considering $F(6)=0$, we have

$$F(x)=\frac{x\,f(x)}{2}-2\int_0^{\tan^{-1}[f(x)/x]}[1+4\cos(\theta)+4\cos^2(\theta)]\,d\theta \quad .$$

Or,

$$F(x)=\frac{x\,f(x)}{2}-2\int_0^{\tan^{-1}[f(x)/x]}[1+4\cos(\theta)+2\cos(2\theta)+2]\,d\theta \ .$$

Hence,

$$F(x)=\frac{x\,f(x)}{2}-2\,[\,3\theta+4\sin(\theta)+\sin(2\theta)]\Big|_0^{\tan^{-1}[f(x)/x]} \ ,$$

$$F(x)=\frac{x\,y}{2}-2\,[\,3\tan^{-1}(\frac{y}{x})+\frac{4y}{\sqrt{x^2+y^2}}+\frac{2xy}{x^2+y^2}].$$

The Algebraic verification of the solution for Example 3 is left to the reader.

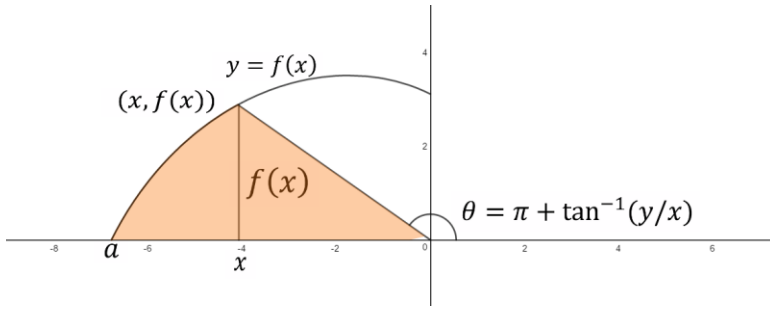
<u>Note that, in the analytical solution for the following example the letter r has been used instead of ρ.</u>

<u>Example 4:</u> Find an anti-derivative for the function $y=f(x)$ implicitly defined by

$$\ln(\frac{x^2+y^2}{2})=\pi+\tan^{-1}(y/x), \quad x\in[-\sqrt{2}\,e^{\pi/2},0),[0,\sqrt{2}\,e^{\pi/4}). \qquad (4)$$

<u>Note that</u>, the point with coordinates $(-\sqrt{2}\,e^{\pi/2},0)$ is an x-intercept of $y=f(x)$.

<u>Analytical Solution:</u> Let $F_1(x)=\int_a^x f(t)dt$ be the area under the graph of $f(x)$ and directly above the interval $[a,x]$, where $a=-\sqrt{2}\,e^{\pi/2}$ is the x-intercept of $f(x)$. Then by Fundamental Theorem of Calculus $F_1(x)$ is an anti-derivative for $f(x)$. Moreover, the coloured area shown in the diagram below can be expressed as $F_1(x)-\frac{xf(x)}{2}$. (Note that for $x\in[-6,-1]$ the triangular part of the coloured area is $-xf(x)/2$, because $x<0$).



On the other hand, with the selection of the polar argument θ as $\theta = \pi + \tan^{-1}(f(x)/x)$, we know from the same coloured polar area is simply $\frac{1}{2}\int_{\pi+\tan^{-1}[f(x)/x]}^{\pi} r(\theta)^2 d\theta$, where $r(\theta)$ is the polar representation of the relation (4), in both quadrants I and II, and hence that of the function $y = f(x)$ as well. Therefore we have

$$F_1(x) - \frac{xf(x)}{2} = \frac{1}{2}\int_{\pi+\tan^{-1}[f(x)/x]}^{\pi} r(\theta)^2 d\theta. \qquad (*)$$

Next, it is easy to check that upon polar substitutions $x = r\cos(\theta)$ and $y = r\sin(\theta)$, and $x^2 + y^2 = r^2$ in (4) the relation is simplified into $r^2 = 2e^{\theta}$. Substituting this in (*) we get,

$$F_1(x) - \frac{xf(x)}{2} = \frac{1}{2}\int_{\pi+\tan^{-1}[f(x)/x]}^{\pi} 2e^{\theta} d\theta = e^{\theta}\Big|_{\pi+\tan^{-1}[f(x)/x]}^{\pi} = e^{\pi} - e^{\pi+\tan^{-1}[f(x)/x]}.$$

Therefore, $F_1(x) = \frac{xf(x)}{2} - e^{\pi+\tan^{-1}[f(x)/x]}$ is an anti-derivative for $y = f(x)$.

<u>Algebraic verification of the solution:</u> It is enough to show that the derivative of the right hand side of $F_1(x) = \frac{xf(x)}{2} - e^{\pi+\tan^{-1}[f(x)/x]}$ is just $f(x)$, so here we go

$$\frac{d}{dx}[\frac{x\ f(x)}{2} - e^{\pi+\tan^{-1}[f(x)/x]}] = \frac{1}{2}(f(x) + x\ f'(x)) - \frac{x\ f'(x) - f(x)}{x^2 + f(x)^2} e^{\pi+\tan^{-1}[f(x)/x]}.$$

On the other hand, since (4) implies $(x^2 + f(x)^2) = 2e^{\pi+\tan^{-1}[f(x)/x]}$, the above right hand side is simplifies into

$$\frac{1}{2}(f(x) + x\ f'(x)) - \frac{(x\ f'(x) - f(x)}{2} == \frac{1}{2}(f(x) + x\ f'(x)) - \frac{1}{2}(x\ f'(x) - f(x)) = f(x)$$

$$F(x) = \frac{x\ f(x)}{2} - e^{\pi+\tan^{-1}[f(x)/x]} + C,$$

Having observed the efficiency of the method through above examples (as well as the exercises I have laid out at the end of the article), it would be desirable also to bring the indefinite integral version of Theorem 1 as follows, whose proof is given regardless of a graph, and any specific real numbers involved a .

Remark: Note that my method of Implicit Integration can be interpreted in the language of Differential Equations, and I have preferred to explain this in a separate article (see article #8 on this same Calculus 2 section of the website).

Theorem 2 [A. Astaneh] **:** A function $y = f(x)$ defined by an implicit relation $R(x, y) = 0$ on a specific domain is integrable if and only if, when upon substitutions $x = r(\theta)\cos(\theta)$, $y = r(\theta)\sin(\theta)$ the implicit relation is converted into polar form $\rho = r(\theta)$, the square polar function $\rho^2 = r(\theta)^2$ is integrable in terms of θ. Moreover, we have

$$\int f(x)dx = \frac{xf(x)}{2} - \frac{1}{2}\int r(\theta)^2 d\theta,$$

where $\theta = tan^{-1}\left[\frac{f(x)}{x}\right]$.

Proof Assuming $\rho = r(\theta)$ represents the polar version of the implicitly defined function $y = f(x)$ upon popular polar substitutions $x = r(\theta)\cos(\theta)$ and $y = r(\theta)\sin(\theta)$, a normal integral such as the left hand side $\int f(x)dx$ above can be first converted into,

$$\int f(x)dx = \int ydx = \int r(\theta)\sin(\theta)\, d[r(\theta)\cos(\theta)] = \int \rho\sin(\theta)\, d[\rho\cos(\theta)]$$

$$= \int \rho\sin(\theta)[\rho'\cos(\theta) - \rho\sin(\theta)]d\theta = \int \rho\rho'\sin(\theta)\cos(\theta)d(\theta) - \int \rho^2\sin^2(\theta)d\theta$$

$$= \frac{1}{2}\int \rho\rho'\sin(2\theta)d(\theta) - \frac{1}{2}\int \rho^2[1-\cos(2\theta)]d\theta \qquad \text{(i)}$$

Next, an application of the method of integration by parts applied to the first integral on the right hand side of (i) ; with $U = \sin(2\theta)$ and $dV = \rho\rho' d\theta$, will show that the first integral is the same as

$$\frac{1}{2}\int \rho\rho'\sin(2\theta)d(\theta) = \frac{1}{4}\rho^2\sin(2\theta) - \frac{1}{2}\int \rho^2\cos(2\theta)d\theta \quad .$$

Therefore

$$\int f(x)dx = \frac{1}{4}\rho^2\sin(2\theta) - \frac{1}{2}\int \rho^2\cos(2\theta)d\theta - \frac{1}{2}\int \rho^2[1-\cos(2\theta)]d\theta$$

$= \frac{2}{4}(\rho \sin\theta)(\rho\cos\theta) - \frac{1}{2}\int \rho^2 d\theta$= $\int f(x)dx = \frac{xf(x)}{2} - \frac{1}{2}\int r(\theta)^2 d\theta$. (ii)

The above integral equation (ii) also implies $y = f(x)$ is integrable if and only if $\rho^2 = r(\theta)^2$ is, and the proof is complete.

Application 1: Integrate the function $y = \sqrt{1 - x^2 + \sqrt{1+x^2}}$, $0 < |x| < \sqrt{3}$
(a seemingly unlikely integrable function by any known method)

Solution: As the reader can verify that upon substitutions $x = \rho\cos\theta$ and $x = \rho \sin\theta$ $x = r(\theta)\cos(\theta)$ the polar representation of the relation $y = \sqrt{1 - x^2 + \sqrt{1+x^2}}$ will be $\rho^2(\theta) = 2 + \cos^2\theta$, which happens to be integrable with respect to θ , so the Theorem applies best. Since from elementary trigonometry integration we also know that

$$\int (2 + \cos^2\theta)\, d\theta = \frac{5}{2}\theta + \frac{1}{4}\sin 2\theta = \frac{5}{2}\theta + \frac{1}{2}\sin\theta\cos\theta,$$

The theorem implies

$$\int \sqrt{1 - x^2 + \sqrt{1+x^2}}\, dx = \int y\, dx = \frac{xy}{2} - \frac{1}{2}\int \rho^2 d\theta =$$

$$\frac{xy}{2} - \frac{1}{2}[\frac{5}{2}\theta + \frac{1}{2}\sin\theta\cos\theta] + C = \frac{xy}{2} - \frac{1}{2}[\frac{5}{2}\tan^{-1}\frac{y}{x} + \frac{xy}{2(x^2+y^2)}] + C$$

$$= \frac{xy}{2}[1 - \frac{1}{2(1+\sqrt{1+x^2})}] - \frac{5}{4}\tan^{-1}\frac{y}{x} + C$$

Where $y = \sqrt{1 - x^2 + \sqrt{1+x^2}}$, which means,

$$\int \sqrt{1 - x^2 + \sqrt{1+x^2}}\ dx =$$

$$\frac{x\sqrt{1 - x^2 + \sqrt{1+x^2}}}{2}[1 - \frac{1}{2(1+\sqrt{1+x^2})}] - \frac{5}{4}\tan^{-1}\frac{\sqrt{1 - x^2 + \sqrt{1+x^2}}}{x} + C.$$

The following application can be considered as a generalization of Application 1.

Application 2 (A two parameter family of integrals): Let $a, b > 0$, and consider the problem of integrating the function

$$y=\sqrt{a-x^2+\sqrt{a^2+bx^2}}\ ,\quad 0\le x\le a$$

and recall that, the method of implicit integration simply says if $r=\rho(\theta)$ is the polar representation of the relation $y=\sqrt{a-x^2+\sqrt{a^2+bx^2}}$, then

$$\int f(x)dx=\frac{xf(x)}{2}-\frac{1}{2}\int\rho(\theta)^2d\theta .$$

Since presently the polar representation of the relation $y=\sqrt{a-x^2+\sqrt{a^2+bx^2}}$ can be verified to be $\rho^2(\theta)=(2a+\frac{b}{2})+\frac{b}{2}\cos 2\theta$, and since

$$\int[(2a+\frac{b}{2})+\frac{b}{2}\cos 2\theta]\,d\theta=(2a+\frac{b}{2})\theta+\frac{b}{4}\sin 2\theta=(2a+\frac{b}{2})\theta+\frac{b}{2}\sin\theta\cos\theta ,$$

the integral relation (1) implies

$$\int\sqrt{a-x^2+\sqrt{a^2+bx^2}}\,dx=\int y\,dx=\frac{xy}{2}-\frac{1}{2}\int\rho^2d\theta=$$

$$\frac{xy}{2}-\frac{1}{2}[(2a+\frac{b}{2})\theta+\frac{b}{2}\sin\theta\cos\theta]+C=\frac{xy}{2}-\frac{1}{2}[(2a+\frac{b}{2})\tan^{-1}\frac{y}{x}+\frac{bxy}{2(x^2+y^2)}]+C$$

$$=\frac{xy}{2}[1-\frac{b}{2(x^2+y^2)}]-(a+\frac{b}{4})\tan^{-1}\frac{y}{x}+C$$

where $y=\sqrt{a-x^2+\sqrt{a^2+bx^2}}$, which means,

$$\int\sqrt{a-x^2+\sqrt{a^2+bx^2}}\,dx=$$

$$\frac{x\sqrt{a-x^2+\sqrt{a^2+bx^2}}}{2}[1-\frac{b}{2(a+\sqrt{a^2+bx^2})}]-(a+\frac{b}{4})\tan^{-1}\frac{\sqrt{a-x^2+\sqrt{a^2+bx^2}}}{x}+C$$

Application 3 (El Ganzo 2016)**:** Integrate the function

$y=f(x)=x^{3/2}\sqrt{x+\sqrt{x^2+2}}$.

(Another function unlikely to be integrated by other techniques)

Solution: This time the polar representation of the function can be written as $\rho^2(\theta)=\frac{1}{2}\tan^4\theta$, and therefore the Theorem implies

$$\int x^{3/2}\sqrt{x+\sqrt{x^2+2}}\,dx = \int y\,dx = \frac{xy}{2} - \frac{1}{2}\int \rho^2 d\theta =$$

$$\frac{xy}{2} - \frac{1}{4}\int \tan^4\theta\, d\theta = \frac{xy}{2} - \frac{1}{4}[\theta - \tan\theta + \frac{1}{3}\tan^3\theta] + C =$$

$$\frac{xy}{2} - \frac{1}{4}[\tan^{-1}\frac{y}{x} - \frac{y}{x} + \frac{y^3}{3x^3}] + C$$

$$\frac{xy}{2} + \frac{y}{4x} - \frac{y^3}{12x^3} - \frac{1}{4}\tan^{-1}\frac{y}{x} + C$$

Where $y = x^{3/2}\sqrt{x+\sqrt{x^2+2}}$. That is,

$$\int x^{3/2}\sqrt{x+\sqrt{x^2+2}}\,dx =$$

$$\frac{1}{2}x^{5/2}\sqrt{x+\sqrt{x^2+2}} - \frac{1}{4}x^{1/2}\sqrt{x+\sqrt{x^2+2}} + \frac{1}{12}x^{3/2}(x+\sqrt{x^2+2})\sqrt{x+\sqrt{x^2+2}}\,.$$

$$+\frac{1}{4}\tan^{-1}(x^{1/2}\sqrt{x+\sqrt{x^2+2}}) + C$$

Note that, in above, to evaluate $\int \tan^4\theta\, d\theta$, upon substitution $u = \tan\theta$, we have $du = (1+\tan^2\theta)d\theta = (1+u^2)d\theta$, $d\theta = \frac{du}{1+u^2}$, and therefore

$$\int \tan^4\theta\, d\theta = \int \frac{u^4}{1+u^2}du = \int [u^2 - 1 + \frac{1}{1+u^2}]du = \frac{1}{3}u^3 - u + \tan^{-1}u =$$

$$\frac{1}{3}\tan^3\theta - \tan\theta + \theta = \frac{y^3}{3x^3} - \frac{y}{x} + \tan^{-1}\frac{y}{x}.$$

Application 4: Consider the problem of integrating the function

$$f(x) = \sqrt{a^2 - x^2}\,, \quad |x| \le a\,.$$

This is usually accomplished by the conventional substitutions $x = a\sin\theta$ or $x = a\sin\theta$, but our Theorem's integration formula offers a more clean cut that any of those substitutions. Consider the polar representation $\rho(\theta) = a$ of the circle representing $y = \sqrt{a^2 - x^2}$, then $\int f(x)dx = \frac{xf(x)}{2} - \frac{1}{2}\int \rho(\theta)^2 d\theta$ implies

$$\int \sqrt{a^2 - x^2}\,dx = \frac{x\sqrt{a^2 - x^2}}{2} - \frac{1}{2}\int r(\theta)^2 d\theta =$$

$$= \frac{x\sqrt{a^2 - x^2}}{2} - \frac{1}{2}\int a^2 \times d\theta = \frac{x\sqrt{a^2 - x^2}}{2} - \frac{1}{2}a^2\theta + C\,,$$

$$= \frac{x\sqrt{a^2 - x^2}}{2} - \frac{a^2}{2}\cos^{-1}\frac{x}{a} + C.$$

The above answer may seem a bit different from the common solution seen for the indefinite integral $\int \sqrt{a^2 - x^2}\,dx$ in literature, as

$$\int \sqrt{a^2 - x^2}\,dx = \frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C.$$

But that is only because of the identity $\sin^{-1} X + \cos^{-1} X = \pi/2$, the two answers are actually the same.

Application 5: (A first sequence of implicitely integrable functions)

Let $n = 2,3,...$ be a positive integer, and consider the problem of integrating the function

$$y = \sqrt{x^{\frac{2n}{n-1}} - x^2}\,, \quad 1 \le x$$

and recall that, the method of implicit integration simply says if $r = \rho(\theta)$ is the polar representation of the relation $y = \sqrt{x^{\frac{2n}{n-1}} - x^2}$, then

$$\int f(x)dx = \frac{xf(x)}{2} - \frac{1}{2}\int \rho(\theta)^2 d\theta. \qquad (1)$$

Since presently the polar representation of the relation $y = \sqrt{x^{\frac{2n}{n-1}} - x^2}$ can be verified to be $\rho = \sec^n \theta$, and since a routine substitution $u = \tan\theta$ to integrate the polar function $\rho^2 = \sec^{2n}\theta$ implies

$$\int \sec^{2n}\theta\, d\theta = \int (1+u^2)^{n-1} du = \int \sum_{k=0}^{n-1} \binom{n-1}{k} u^{2k} du = \sum_{k=0}^{n-1} \frac{1}{2k+1}\binom{n-1}{k} \tan^{2k+1}\theta\,,$$

the integral relation (1) will imply,

$$\int\sqrt{x^{\frac{2n}{n-1}}-x^2}\,dx=\int y\,dx=\frac{xy}{2}-\frac{1}{2}\int\rho(\theta)^2d\theta=\frac{xy}{2}-\frac{1}{2}\sum_{k=0}^{n-1}\frac{1}{2k+1}\binom{n-1}{k}\tan^{2k+1}\theta+C$$

$$\frac{xy}{2}-\frac{1}{2}\sum_{k=0}^{n-1}\frac{1}{2k+1}\binom{n-1}{k}(\frac{y}{x})^{2k+1}+C$$

where $y=\sqrt{x^{\frac{2n}{n-1}}-x^2}$, which means,

$$\int\sqrt{x^{\frac{2n}{n-1}}-x^2}\,dx=\frac{1}{2}[x\sqrt{x^{\frac{2n}{n-1}}-x^2}-\sum_{k=0}^{n-1}\frac{1}{2k+1}\binom{n-1}{k}(\frac{\sqrt{x^{\frac{2n}{n-1}}-x^2}}{x})^{2k+1}]+\mathrm{C}.$$

<u>For example:</u> , when $n=4$, we have

$$\int\sqrt{x^{\frac{8}{3}}-x^2}\,dx=\frac{1}{2}[x\sqrt{x^{\frac{8}{3}}-x^2}-\sum_{k=0}^{3}\frac{1}{2k+1}\binom{3}{k}(\frac{\sqrt{x^{\frac{8}{3}}-x^2}}{x})^{2k+1}]+\mathrm{C}=$$

$$\frac{1}{2}[x\sqrt{x^{\frac{8}{3}}-x^2}-\frac{\sqrt{x^{\frac{8}{3}}-x^2}}{x}-(\frac{\sqrt{x^{\frac{8}{3}}-x^2}}{x})^3-\frac{3}{5}(\frac{\sqrt{x^{\frac{8}{3}}-x^2}}{x})^5-\frac{1}{7}(\frac{\sqrt{x^{\frac{8}{3}}-x^2}}{x})^7]+C$$

<u>Application 6:</u> (<u>A second sequence of implicitly integrable functions)</u>

Let $n=1,2,\ldots$ be a positive integer, and consider the problem of integrating the function

$$y=\sqrt{x^{\frac{2n}{n+2}}-x^2},\quad 1\le x$$

and recall that, the method of implicit integration simply says if $r=\rho(\theta)$ is the polar representation of the relation $y=\sqrt{x^{\frac{2n}{n+2}}-x^2}$, then

$$\int f(x)dx=\frac{xf(x)}{2}-\frac{1}{2}\int\rho(\theta)^2d\theta.\qquad(1)$$

Since presently the polar representation of the relation $y=\sqrt{x^{\frac{2n}{n+2}}-x^2}$ can be verified to be $\rho^2=\cos^n\theta$, the integral relation (1) will imply,

$$\int\sqrt{x^{\frac{2n}{n+2}}-x^2}\,dx=\int y\,dx=\frac{xy}{2}-\frac{1}{2}\int\rho(\theta)^2d\theta=\frac{xy}{2}-\frac{1}{2}\int\cos^n\theta\,d\theta$$

<u>For example:</u> **<u>(a)</u>** When $n=1$, we have $\rho^2=\cos\theta$ and

$$\int\sqrt{x^{\frac{2}{3}}-x^2}\,dx=\frac{xy}{2}-\frac{1}{2}\int\cos\theta\,d\theta=\frac{xy}{2}-\frac{1}{2}\sin\theta+C=\frac{xy}{2}-\frac{y}{2\sqrt{x^2+y^2}}+C$$

where $y=\sqrt{x^{\frac{2}{3}}-x^2}$, which means,

$$\int\sqrt{x^{\frac{2}{3}}-x^2}\,dx=\frac{x\sqrt{x^{\frac{2}{3}}-x^2}}{2}-\frac{\sqrt{x^{\frac{2}{3}}-x^2}}{2x^{\frac{1}{3}}}+C=\frac{(x^{\frac{4}{3}}-1)\sqrt{x^{\frac{2}{3}}-x^2}}{2x^{\frac{1}{3}}}+C\,.$$

(b) When $n=3$, we have $\rho^2=\cos^3\theta=\frac{1}{4}(\cos3\theta+3\cos\theta)$ and

$$\int\sqrt{x^{\frac{6}{5}}-x^2}\,dx=\frac{xy}{2}-\frac{1}{2}\int\cos^3\theta\,d\theta=\frac{xy}{2}-\frac{1}{8}\int(\cos3\theta+3\cos\theta)\,d\theta=$$

$$\frac{xy}{2}-\frac{1}{8}[\frac{1}{3}\sin3\theta+3\sin\theta]+C=\frac{xy}{2}-\frac{1}{8}[\frac{1}{3}(3\sin\theta-4\sin^3\theta)+3\sin\theta]+C=$$

$$\frac{xy}{2}-\frac{1}{2}[\sin\theta-\frac{1}{3}\sin^3\theta]+C=\frac{xy}{2}-\frac{1}{2}\sin\theta[1-\frac{1}{3}\sin^2\theta]=$$

$$\frac{xy}{2}-\frac{1}{6}\sin\theta[2+\cos^2\theta]+C=\frac{xy}{2}-\frac{y}{6\sqrt{x^2+y^2}}[2+\frac{x^2}{(x^2+y^2)}]+C$$

where $y=\sqrt{x^{\frac{6}{5}}-x^2}$, which means,

$$\int\sqrt{x^{\frac{6}{5}}-x^2}\,dx=\frac{x\sqrt{x^{\frac{6}{5}}-x^2}}{2}-\frac{\sqrt{1-x^{\frac{4}{5}}}}{3}-\frac{x^{\frac{4}{5}}\sqrt{1-x^{\frac{4}{5}}}}{6}+C$$

Application 7: (A third sequence of implicitly integrable functions)

Let $n=0,1,2,\ldots$ be a positive integer, and consider the problem of integrating the function

$$y=x\sqrt{x^{\frac{4}{2n-1}}-1}\,,$$

and recall that, the method of implicit integration simply says if $r=\rho(\theta)$ is the polar representation of the relation $y=x\sqrt{x^{\frac{4}{2n-1}}-1}$, then

$$\int f(x)dx=\frac{xf(x)}{2}-\frac{1}{2}\int\rho(\theta)^2d\theta\,. \qquad (1)$$

Since presently the polar representation of the relation $y=x\sqrt{x^{\frac{4}{2n-1}}-1}$ can be verified to be $\rho^2=\sec^{2n+1}\theta$, the integral relation (1) will imply,

$$\int x\sqrt{x^{\frac{4}{2n-1}}-1}\,dx=\int y\,dx=\frac{xy}{2}-\frac{1}{2}\int\rho(\theta)^2d\theta=\frac{xy}{2}-\frac{1}{2}\int\sec^{2n+1}\theta\,d\theta+\mathrm{C}.$$

For example , (**a**) When $n=0$, we have $y=x\sqrt{x^{-4}-1}=\sqrt{x^{-2}-x^2}=\frac{\sqrt{1-x^4}}{x}$ and $\rho^2=\sec\theta$, therefore the integral relation (1) implies,

$$\int\frac{\sqrt{1-x^4}}{x}\,dx=\frac{xy}{2}-\frac{1}{2}\int\sec\theta\,d\theta=\frac{xy}{2}-\frac{1}{2}\ln(\sec\theta+\tan\theta)+C=$$

$$\frac{xy}{2}-\frac{1}{2}\ln(\frac{\sqrt{x^2+y^2}}{x}+\frac{y}{x})+C$$

where $y=\frac{\sqrt{1-x^4}}{x}$. Since $\sqrt{x^2+y^2}=\sqrt{x^2+\frac{1-x^4}{x^2}}=\frac{1}{x}$ we have,

$$\int\frac{\sqrt{1-x^4}}{x}\,dx=\frac{\sqrt{1-x^4}}{2}-\frac{1}{2}\ln(\frac{1}{x^2}+\frac{\sqrt{1-x^4}}{x^2})+C=$$

$$\frac{1}{2}[\sqrt{1-x^4}-\ln(1+\sqrt{1-x^4})+\ln(x^2)]+C$$

(**b**), when $n=1$, we have $y=x\sqrt{x^4-1}$,

$$\int x\sqrt{x^4-1}\,dx=\frac{xy}{2}-\frac{1}{2}\int\sec^3\theta\,d\theta=\frac{xy}{2}-\frac{1}{2}\times\frac{1}{2}[\sec\theta\tan\theta+\ln(\sec\theta+\tan\theta)]+C=$$

$$\frac{xy}{2}-\frac{1}{4}[\frac{(\sqrt{x^2+y^2})y}{x^2}+\ln(\frac{\sqrt{x^2+y^2}}{x}+\frac{y}{x})]+C$$

where $y=x\sqrt{x^4-1}$, which means,

$$\int x\sqrt{x^4-1}\,dx=\frac{1}{4}[2x^2\sqrt{x^4-1}-x^2\sqrt{x^4-1}-\ln(x^2+\sqrt{x^4-1})]+C=$$

$$\frac{1}{4}[x^2\sqrt{x^4-1}-\ln(x^2+\sqrt{x^4-1})]+C$$

So far, in dealing with the method of implicit integration we have been exclusively concerned with finding solutions to challenging Cartesian integrals by converting them into manageable trigonometric integrals according to the integration formula ,

$$\int y dx=\frac{xy}{2}-\frac{1}{2}\int\rho(\theta)^2d\theta.$$

A second point of view would be that the above integral relation can be expressed in a “reverse” form, that is the polar integral $\int\rho(\theta)^2d\theta$ on the right hand side can be expressed in terms of the Cartesian integral $\int ydx$ on the left. More precisely, upon substitutions $x = \rho\cos(\theta)$ and $y = \rho\sin(\theta)$ on the right hand side of the above integral formula, and necessary simplifications, the formula can be expressed in the form

$$\int \rho(\theta)^2 d\theta = \frac{1}{2}\rho(\theta)^2 \sin(2\theta) - 2\int ydx.$$

This reversed form can be used to find some challenging trigonometric integrals by reducing them manageable Cartesian integrals $\int ydx$. A typical example is,

Aplication 8: Consider the problem of integrating trigonometric function $\sec^2(\theta)\ln(\cot\theta)$, and set $\rho^2 = \sec^2(\theta)\ \ln(\cot\theta)$. Then, upon substitution of $\sec^2(\theta)\ \ln(\cot\theta)$ for $\rho(\theta)^2$ on both sides of (2) we can get,

$$\int \sec^2\theta \ln(\cot\theta) d\theta = \frac{1}{2}\sec^2\theta\ \ln(\cot\theta)\sin(2\theta) - 2\int ydx.$$

It is therefore enough to find the Cartesian representation $y = f(x)$ of the polar equation $\rho^2 = \sec^2(\theta)\ \ln(\cot\theta)$ first, and then integrate $\int ydx$ it in the right hand side and then use the relation $x = \rho\cos(\theta)$ to get the desired solution.

In order to find the Cartesian form of the equation $\rho^2 = \sec^2(\theta)\ \ln(\cot\theta)$, we only need to substitute $\rho^2 = x^2 + y^2$, $\sec^2\theta = \frac{x^2 + y^2}{x^2}$, and $\cot\theta = \frac{x}{y}$ in the relation. Then we get $x^2 + y^2 = \frac{x^2 + y^2}{x^2}\ \ln\frac{x}{y}$ which can be simplified into $x^2 = \ln\frac{x}{y}$. This in turn implies $y = xe^{-x^2}$, for which $\int ydx = -\frac{1}{2}e^{-x^2}$. Hence ,

$$\int \sec^2\theta \ln(\cot\theta) d\theta = \frac{1}{2}\sec^2\theta\ \ln(\cot\theta)\sin(2\theta) - 2\int xe^{-x^2} dx =$$

$$\tan\theta\ \ln(\cot\theta) + e^{-x^2} + C = \tan\theta \ln(\cot\theta) + e^{-\rho^2 coc^2\theta} + C =$$

$$\tan\theta \ln(\cot\theta) + e^{-\ln(\cot\theta)} + C = \tan\theta \ln(\cot\theta) + (\cot\theta)^{-1} + C =$$

$$\tan\theta \ln(\cot\theta) + \tan\theta + C = \tan\theta\,(\ln(\cot\theta) + \tan\theta) + C.$$

The above solution is confirms by the final answer provided by Wolfram integration tool.

Note that, somewhere in above argument, we have used the fact that $\rho^2 \cos^2(\theta) = \ln(\cot\theta)$, which is only a restatement of the original polar equation $\rho^2 = \sec^2(\theta)\ \ln(\cot\theta)$.

Suggested Exercises

1: Use analytical method, or algebraically verify that for any $a > 0$ the most general anti-derivative for the function $y = f(x)$, on an appropriately selected domain, defined by the implicit relation $(x^2 + y^2)^2 = a^2xy$ is given by

$$F(x) = \frac{xy}{2} - \frac{a^2(x^2 - y^2)}{8(x^2 + y^2)} + C$$

where $y = f(x)$ is the function defined by the original implicit relation.

2: Show that for any $a \neq 0$ the general anti-derivative for the function $y = f(x)$ defined by implicit relation $y = x \tan\left[\,(x^2 + y^2)^a\right]$, on an appropriately selected domain will be,

$$F(x) = \frac{xy}{2} - \frac{a}{2(a+1)} \tan^{-1}\left[\left(\frac{y}{x}\right)^{\frac{a+1}{a}}\right] + C,$$

where $y = f(x)$ is the function defined by the original implicit relation.

3: Given that $n > 2$ is a positive integer, integrate, $\int \frac{\sqrt[n]{\tan^2\theta}}{\cos^2\theta}\, d\theta$.

[Answer , $\int \frac{\sqrt[n]{\tan^2\theta}}{\cos^2\theta}\, d\theta = \frac{n\tan\theta\sqrt[n]{\tan^2\theta}}{(n+2)\cos^2\theta} + C$]

4: Use the method of implicit integration to integrate $\int \frac{d\theta}{(a\sin\theta + b\cos\theta)^2}$.

[Having found the solution as $\int \frac{d\theta}{(a\sin\theta + b\cos\theta)^2} = -\frac{\cos\theta}{a\,(a\sin\theta + b\cos\theta)} + C$, conclude that an appropriate substitution to find the original integral would have been $u = \frac{\cos\theta}{a\sin\theta + b\cos\theta}$ in the first place.]

Now integrate $\int \frac{\cos^n\theta\, d\theta}{(a\sin\theta + b\cos\theta)^{n+2}}$, using the same substitution.

<u>5</u>: Show that, for $f(x)=\sqrt[3]{x[1+\sqrt{1+(x^4/27)}]}+\sqrt[3]{x[1-\sqrt{1+(x^4/27)}]}$,

$$\int f(x)dx=\frac{x\ f(x)}{2}+\ln\left(\frac{\sqrt{(x^2+f(x)^2)}}{\sqrt{2}\ f(x)}\right)+C$$