<u>Altering Corners of Curves for Smoothness; An Algorithm</u>

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I posted this article on the BC Association of Math Teachers' e-mail list sometime around 2002/2003, but apart from that I am publishing it here on my website for the first time. Obviously this would be a more updated one.

The main objective of the article is to present a Calculus Algorithm to replace a segment of a curve which contains a single corner point at $x = a$ with a segment from a parabola in such a way that the newly repaired curve differs with the original one only over an arbitrarily small interval $(a - \mathcal{E}, a + \mathcal{E})$ containing $x = a$. More precisely, assume you have a function defined piecewise such as

$$H(x) = \begin{cases} f(x), & x \le a \\ g(x), & x > a \end{cases} \quad ,$$

which has a corner point (i.e., *H(x)* is non-differentiable at $x = a$**)**, but otherwise *f(a)=g(a)* and for convenience we assume that functions *f(x)* and *g(x)*are both differentiable over their respective domains $(-\infty, a]$ d $[a, \infty)$.

The following process shows how you can replace a segment from the graph of $H(x)$ containing the corner point $(a,\ H(a)\ )$ by a specific segment of a parabola, say represented by $y = A(x-p)^2 + q$, in such a way that the resulting curve represented by a function $H_{(\mathcal{E},\delta)}$ becomes entirely smooth, in the sense its graph having no corner points.

Following the assertions and proofs, two detailed examples will examine the efficiency of the method. As a second part of the article two propositions very closely related to the algorithm are represented, each followed by an example. That means in total we have four examples in this article.

<u>The Alteration Algorithm</u>: Let $H(x)$ be a continuous function defined as above and assume $f(x)$ and $g(x)$ are differentiable over their respective domains $(-\infty, a]$and $[a, \infty)$; however assume that $f'(a) \neq g'(a)$. If there exists a pair of positive numbers (ε, δ) satisfying the equation

$$(\delta + \varepsilon)[g'(a+\varepsilon) + f'(a-\delta)] = 2[g(a+\varepsilon) - f(a-\delta)],$$

then the following steps will present a replacement function

$$H_{(\varepsilon,\delta)}(x) = \begin{cases} f(x), & x \le a-\delta \\ A(x-p)^2+q, & a-\delta < x < a+\varepsilon \\ g(x), & a+\varepsilon \le x \end{cases}$$

for $H(x)$, such that $H_{(\varepsilon,\delta)}(x)$ is differentiable everywhere, and that the two functions differ only on the open interval $(a-\delta,\ a+\varepsilon)$.

Remark In ideal cases, such as the cases of the two examples following the proofs, one of the variables δ or ε will be solved in terms of the other one from the above equation, say δ in terms of ε, and hence for any arbitrarily small positive number ε, the algorithm will find an entirely differentiable replacement function $H_\varepsilon(x)$ that differs from $H(x)$ only over the open interval $(\ a-\delta(\varepsilon)\ ,\ a+\varepsilon\ )$.

Algorithm steps:

Step 1: Solve the equation

$$(\delta+\varepsilon)[g'(a+\varepsilon)+f'(a-\delta)] = 2[g(a+\varepsilon)-f(a-\delta)]$$

for a pair of positive numbers (ε,δ).

Note: If no pair of positive numbers (ε,δ) satisfying the above equation exists, the alteration isn't possible, and the process must be stopped.

Step 2: Having found a pair (ε,δ) in step 1, evaluate

$$p = a - \frac{\delta g'(a+\varepsilon)+\varepsilon f'(a-\delta)}{g'(a+\varepsilon)-f'(a-\delta)}.$$

Step 3: Having obtained p in terms of ε, δ, and a in step 2, evaluate

$$A = \frac{f'(a-\delta)}{2(a-\delta-p)}.$$

Note: If $f(x)$ happens to be a constant function, instead evaluate A as

$$A = \frac{g'(a+\varepsilon)}{2(a+\varepsilon-p)}\ .$$

Step 4: Finally evaluate

$$q = f(a-\delta) - A(a-\delta-p)^2.$$

Step 5: Now consider the parabola $y_{\varepsilon,\delta} = A(x-p)^2 + q$ as the promised patching parabola for $H(x)$, and define the differentialized version $H_{\varepsilon,\delta}(x)$ of $H(x)$ as follows,

$$H_{(\varepsilon,\delta)}(x) = \begin{cases} f(x), & x \le a-\delta \\ A(x-p)^2 + q, & a-\delta < x < a+\varepsilon \\ g(x), & a+\varepsilon \le x \end{cases}$$

The Proof: Assume that the altering parabola is $y = A(x-p)^2 + q$. We first observe that, since the parabola has to be tangent to the graph of $f(x)$ at $x = a-\delta$ and also tangent to the graph of $g(x)$ at $x = a+\varepsilon$, we should have

$$f'(a-\delta) = \frac{d}{dx} A(x-p)^2 + q\Big|_{x=a-\delta} = 2A(a-\delta-p)$$
$$g'(a+\varepsilon) = \frac{d}{dx} A(x-p)^2 + q\Big|_{x=a+\varepsilon} = 2A(a+\varepsilon-p)$$,

therefore

$$A = \frac{f'(a-\delta)}{2(a-\delta-p)} \qquad (1)$$

$$A = \frac{g'(a+\varepsilon)}{2(a+\varepsilon-p)} \qquad (2)$$

These two equations imply

$$(a+\varepsilon-p)f'(a-\delta) = (a-\delta-p)g'(a+\varepsilon) \quad . \qquad (3)$$

If we now solve the equation (3) for p we obtain

$$p = a - \frac{\delta g'(a+\varepsilon) + \varepsilon f'(a-\delta)}{g'(a+\varepsilon) - f'(a-\delta)} \qquad (4)$$

On the other hand, since $y = A(x-p)^2 + q$ must go through both points $(a-\delta, f(a-\delta))$ and $(a+\varepsilon, g(a+\varepsilon))$, we should also have

$$q = f(a-\delta) - A(a-\delta-p)^2 \quad , \qquad (5)$$
$$q = g(a+\varepsilon) - A(a+\varepsilon-p)^2$$

and therefore

$$f(a-\delta) - A(a-\delta-p)^2 = g(a+\varepsilon) - A(a+\varepsilon-p)^2 \qquad (6)$$

Now substituting values for *A* from (1) and (2) in the left and the right hand side of (6) respectively, it follows that

$$2f(a-\delta) - (a-\delta-p)f'(a-\delta) = 2g(a+\varepsilon) - (a+\varepsilon-p)g'(a+\varepsilon),$$

or,

$$2f(a-\delta) - (a-\delta)f'(a-\delta) - 2g(a+\varepsilon) + (a+\varepsilon)g'(a+\varepsilon) = [g'(a+\varepsilon) - f'(a-\delta)]\,p$$

and by (4) the right hand side of this inequality can be written as

$$\begin{aligned}&2f(a-\delta) - (a-\delta)f'(a-\delta) - 2g(a+\varepsilon) + (a+\varepsilon)g'(a+\varepsilon) = \\ &a\,[g'(a+\varepsilon) - f'(a-\delta)] - \delta\; g'(a+\varepsilon) - \varepsilon f'(a-\delta)\end{aligned} .$$

This equation can be put in the form

$$\begin{aligned}&-(a-\delta)f'(a-\delta) + (a+\varepsilon)g'(a+\varepsilon) - (a-\delta)g'(a+\varepsilon) + (a+\varepsilon)f'(a-\delta) \\ &= 2g(a+\varepsilon) - 2f(a-\delta)\end{aligned} .$$

And this equation in turn can be simplified into

$$(\varepsilon+\delta)\;[\,g'(a+\varepsilon) + f'(a-\delta)\,] \;=\; 2[g(a+\varepsilon) - f(a-\delta)]. \qquad (7)$$

Equation (7) can be used first to express δ in terms of a and ε; or ε in terms of a and δ, whichever is easier. Then, say having obtained δ in terms of ε, one can use (4) to find *p* in terms of ε. Then *A* can be obtained from any of (1) or (2). And finally any of the equations in (5) will present *q*.

<u>Example 1</u> Let $H(x) = \begin{cases} x^2, & x \le 0 \\ \sqrt{x}, & x > 0 \end{cases}$

,

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> f := x -> piecewise(x <0,x^2, x<12, sqrt(x));
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$$f := x \to \text{piecewise}(x < 0, x^2, x < 12, \sqrt{x})$$

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plot(f(x), x=-1..1);
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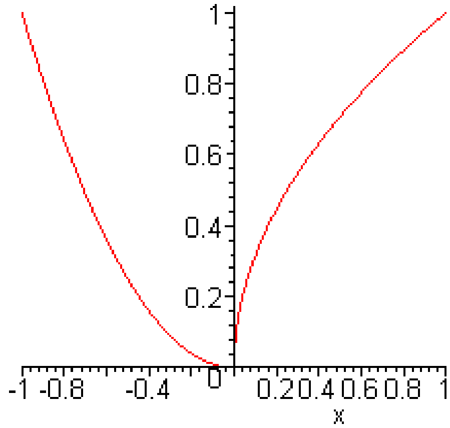

In order to obtain an alteration we follow the above steps of the algorithm.

Step 1: Considering that $f(x)=x^2$ and $g(x)=\sqrt{x}$, and $a=0$, the equation

$(\delta+\varepsilon)[g'(a+\varepsilon)+f'(a-\delta)]=2[g(a+\varepsilon)-f(a-\delta)]$ will read as

$(\delta+\varepsilon)[\frac{1}{2\sqrt{\varepsilon}}-2\delta]=2[\sqrt{\varepsilon}-\delta^2]$, from which we get $\delta=\frac{3\varepsilon}{1-4\varepsilon\sqrt{\varepsilon}}$.

Step 2:

$$p=a-\frac{\delta g'(a+\varepsilon)+\varepsilon f'(a-\delta)}{g'(a+\varepsilon)-f'(a-\delta)}=\frac{-3\varepsilon(1-4\varepsilon\sqrt{\varepsilon})}{1+8\varepsilon\sqrt{\varepsilon}}.$$

Step 3:

$$A=\frac{f'(a-\delta)}{2(a-\delta-p)}=\frac{1+8\varepsilon\sqrt{\varepsilon}}{16\varepsilon\sqrt{\varepsilon}(1-\varepsilon\sqrt{\varepsilon})}.$$

Step 4:

$$q=f(a-\delta)-A(a-\delta-p)^2=\frac{9\varepsilon^2}{1+8\varepsilon\sqrt{\varepsilon}}.$$

Step 5: Therefore, for any $0<\varepsilon$ the following alteration $H_\varepsilon(x)$ is obtained.

$$H_\varepsilon(x)=\begin{cases} x^2, & x\le -\dfrac{3\varepsilon}{1-4\varepsilon\sqrt{\varepsilon}} \\ A(x-p)^2+q, & -\dfrac{3\varepsilon}{1-4\varepsilon\sqrt{\varepsilon}}<x<\varepsilon \\ \sqrt{x}, & \varepsilon\le x \end{cases}$$

where $A = \dfrac{1+8\varepsilon\sqrt{\varepsilon}}{16\varepsilon\sqrt{\varepsilon}(1-\varepsilon\sqrt{\varepsilon})}$, $p = \dfrac{-3\varepsilon(1-4\varepsilon\sqrt{\varepsilon})}{1+8\varepsilon\sqrt{\varepsilon}}$, and $q = \dfrac{9\varepsilon^2}{1+8\varepsilon\sqrt{\varepsilon}}$.

In particular, choosing $\varepsilon = \frac{1}{4}$, we get $\delta = \frac{3}{2}$, $p = -\frac{3}{16}$, $A = \frac{8}{7}$, and $q = \frac{9}{32}$.

Therefore, one of the infinitely many examples of an alteration for $H(x)$ is given by

$$H_{\frac{1}{4}}(x) = \begin{cases} x^2, & x \le -\dfrac{3}{2} \\ \dfrac{8}{7}(x+\dfrac{3}{16})^2 + \dfrac{9}{32}, & -\dfrac{3}{2} < x < \dfrac{1}{4} \\ \sqrt{x}, & \dfrac{1}{4} \le x \end{cases}$$

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> f := x -> piecewise(x <(-3/2),x^2,
x<1/4,(8/7)*(x+(3/16))^2+(9/32),x<12,sqrt(x));
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$$f := x \to \text{piecewise}\left(x < \frac{-3}{2}, x^2, x < \frac{1}{4}, \frac{8}{7}\left(x+\frac{3}{16}\right)^2 + \frac{9}{32}, x < 12, \sqrt{x}\right)$$

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> plot(f(x), x=-4..4);
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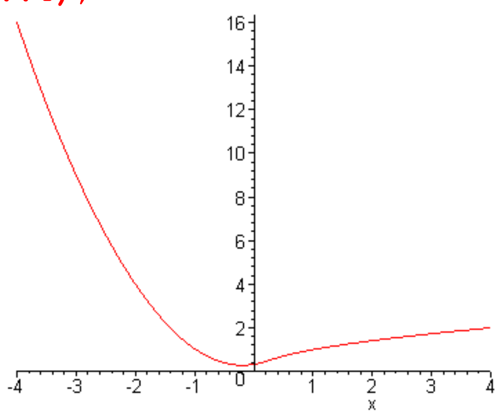

Example 2 Consider the function $H(x) = \begin{cases} 1 - \frac{1}{4}(x-6)^2, & x \le 8 \\ 0 & , x > 8 \end{cases}$,

and its graph given below.

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> f := x -> piecewise( x<8, 1-(1/4)*(x-6)^2, x<12,0 );
```

$$f := x \to \text{piecewise}\left(x < 8, 1 - \frac{1}{4}(x-6)^2, x < 12, 0\right)$$

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> plot(f(x), x=-1..10);
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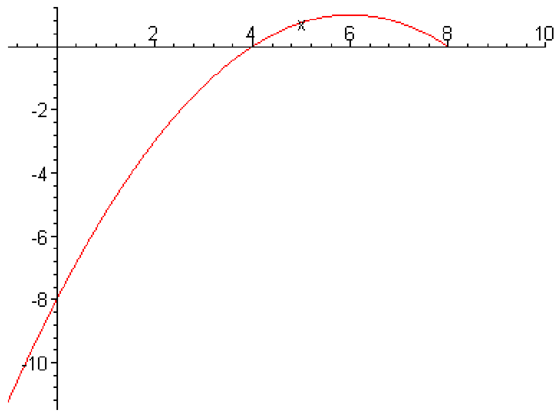


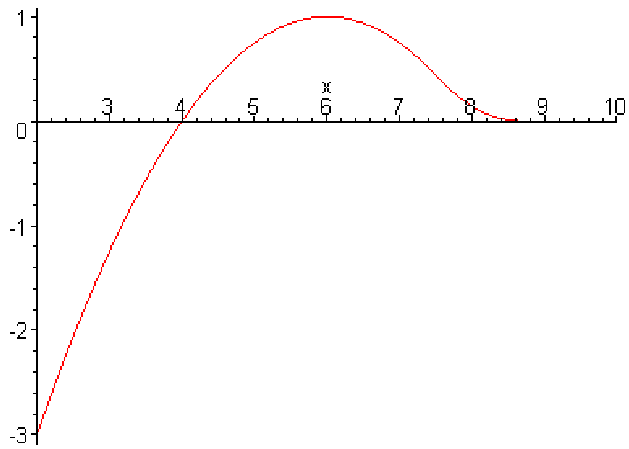
Note that the graph consists of part of the parabola as far as its *x-intercept* at *x=8*, patched with the part of the *x-axis* to the right of *x=8*.
Then the algorithm presents the following family of functions $\{H_\varepsilon(x):\ 0<\varepsilon\}$, each of whose members is an alteration for $H(x)$. For any arbitrary $0<\varepsilon$, and hence the following function $H_\varepsilon(x)$ is entirely differentiable, and only differs from $H(x)$ on the tiny little open interval $(8-\delta,\ 8+\varepsilon)$, with $\delta=\frac{2\varepsilon}{2+\varepsilon}$. Therefore,

$$H_\varepsilon(x)=\begin{cases} 1-\frac{1}{4}(x-6)^2, & x\le 8-\frac{2\varepsilon}{2+\varepsilon} \\ \frac{1}{\varepsilon(\varepsilon+4)}(x-8-\varepsilon)^2, & 8-\frac{2\varepsilon}{2+\varepsilon}<x\le 8+\varepsilon \\ 0, & 8+\ \varepsilon<x \end{cases}$$

Setting $\varepsilon=\frac{1}{2}$, one obtains the following specific alteration for $H(x)$;

$$H_{\frac{1}{2}}(x)=\begin{cases} 1-\frac{1}{4}(x-6)^2, & x\le 8-\frac{2}{5} \\ \frac{4}{9}(x-8-\frac{1}{2})^2, & 8-\frac{2}{5}<x\le 8+\frac{1}{2} \\ 0, & 8+\frac{1}{2}<x \end{cases}$$

And the following is how the graph of the above alteration looks like,



I ccontinue the article with two related propositions, each followed by an example.

Proposition1 If $\frac{d}{dx}\ \ln\ [g(x) - f(-x)]\Big|_{x=\varepsilon} = \frac{1}{\varepsilon}$ for any $0 < \varepsilon$, then the function

$$H(x) = \begin{cases} f(x), & x \le 0 \\ g(x), & x > 0 \end{cases}$$

has an smooth alternation $H_{\varepsilon}(x)$ which differs from $H(x)$ only over the open interval $(-\varepsilon,\ \varepsilon)$.

Proof: It is enough to consider that for $a = 0$ and $\delta = \varepsilon$ equation (7), that is

$$(\delta + \varepsilon)[g'(\varepsilon) + f'(-\delta)] = 2[g(\varepsilon) - f(-\delta)]\ ,$$

can be expressed as

$$\frac{d}{dx}\ \ln\ [g(x) - f(-x)]\Big|_{x=\varepsilon} = \frac{1}{\varepsilon}.$$

The above proposition shows that for the following category of piecewise functions the smooth alteration is possible over any arbitrarily small interval $(-\varepsilon,\ \varepsilon)$.

Example 3 Consider the function $H(x) = \begin{cases} kx^2, & x \le 0 \\ kx^2 + x, & x > 0 \end{cases}$, where $k \neq 0$ is arbitrary. Then for any arbitrarily small $0 < \varepsilon$, the following is an alteration for $H(x)$ whose graph differs from that of $H(x)$ only over the open interval $(-\varepsilon,\ \varepsilon)$.

$$H_{\varepsilon}(x) = \begin{cases} kx^2, & x \leq -\varepsilon \\ \left(k + \frac{1}{4\varepsilon}\right)[x + \frac{\varepsilon}{4k\varepsilon + 1}]^2 + \frac{k\varepsilon^2}{4k\varepsilon + 1}, & -\varepsilon < x < \varepsilon \\ kx^2, & \varepsilon \leq x \end{cases}$$

Proposition 2 Let $H(x)$ be an even continuous function, which is only non-differetiable at $x = 0$. Then for any $0 < \varepsilon$ there exists an entirely smooth even alteration $H_{\varepsilon}(x)$, which differs from $H(x)$ only over the interval $(-\varepsilon,\ \varepsilon)$.

Proof: It is enough to consider that for $a = 0$ and $\delta = \varepsilon$ equation (7) is written as

$$\varepsilon[H'(\varepsilon) - H'(-\varepsilon)] = [H(\varepsilon) - H(-\varepsilon)]\ ,$$

and that, since $H(x)$ is even, the square brackets on both sides are zero. Therefore for any $0 < \varepsilon$ the pair (ε,δ) with $\delta = \varepsilon$ is a solution of (7), and the result follows.

The following example shows the above proposition can be extended to even functions that have an infinite discontinuity.

Example 4 Let $H(x) = \dfrac{1}{x^2}$. Then for any $0 < \varepsilon$ the function

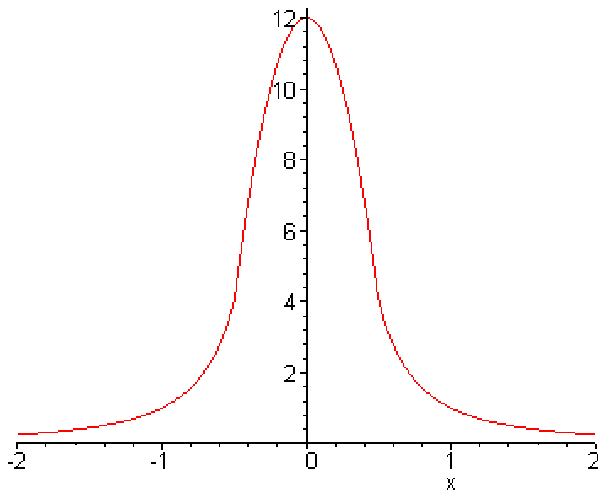
$$H_{\varepsilon}(x) = \begin{cases} \frac{1}{x^2}, & x \leq -\varepsilon \\ -\frac{2}{\varepsilon^4}x^2 + \frac{3}{\varepsilon^2}, & -\varepsilon < x < \varepsilon \\ \frac{1}{x^2}, & \varepsilon \leq x \end{cases}$$

is an entirely differentiable alteration for $H(x)$, whose graph differs from that of $H(x)$ only over the open interval $(-\varepsilon,\ \varepsilon)$. For $\varepsilon = \frac{1}{2}$ the following shows the graph of $H_{\frac{1}{2}}(x)$.

```
> f := x -> piecewise(x <-0.5,1/(x^2), x<0.5,-32*x^2+12,
x<12, 1/(x^2));
>
>
```

$$f := x \rightarrow \text{piecewise}\left(x < -.5, \frac{1}{x^2}, x < .5, -32\, x^2 + 12, x < 12, \frac{1}{x^2}\right)$$

- plot(f(x), x=-2..2);
-



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