A Calculus Proposition and an Application

Ali Astaneh, Vancouver, B.C

Proposition: Let $f(x)$ be a differentiable function and $A(a, f(a))$, $B(b, f(b))$ two points on its graph, and let Ta, Tb denote tangent lines at A and B respectively.

Then for $0 < s < 1$ the following two conditions are equivalent:

(a) $\dfrac{f(b) - f(a)}{b-a} = (1-s)f'(a) + s\ f'(b)$

(b) $c = s\,a + (1-s)\,b$ is the abscissa of the point of intersection of Ta and Tb.

Note, that this means the same linear combination expressing the slope of the secant segment AB in terms of the respective slopes $f'(a)$ and $f'(b)$ of Ta and Tb will determine how the abscissa of the point of inter section of the lines Ta and Tb can be expressed in terms of a and b, by switching the coefficients of the linear combination.

Proof: (a) → (b) Let c be the abscissa of the point of intersection of Ta and Tb. Since $f'(a) = \frac{f(c)-f(a)}{c-a}$ and $f'(b) = \frac{f(c)-f(b)}{c-b}$, eliminating $f(c)$ from these two relations, and solving the resulting relation for c implies

$$c = \frac{bf'(b) - af'(a) - f(b) + f(a)}{f'(b) - f'(a)}. \qquad (*)$$

On the other hand the assumption $\dfrac{f(b) - f(a)}{b-a} = (1-s)f'(a) + s\ f'(b)$ in part (a) can be written as $-f(b) + f(a) = -(b-a)[(1-s)f'(a) + sf'(b)]$. If you now substitute the right hand side of this latter relation for $-f(b) + f(a)$ in the numerator of the fraction (*) and simplify the fraction (patiently!) , you get exactly $c = s\,a + (1-s)\,b$.

To prove (a) → (b) assume that the abscissa of the point of intersection is $c = s\,a + (1-s)\,b$. Since the same abscissa should be given by (*) we have relation,

$$c = s\,a + (1-s)\,b$$

$$c = \frac{bf'(b) - af'(a) - f(b) + f(a)}{f'(b) - f'(a)}.$$

Now, equating the two right hand sides of the above, the resulting relation can (patiently!) be expressed as

$$\frac{f(b) - f(a)}{b-a} = (1-s)\,f'(a) + sf'(b).$$

And the proof of the proposition is complete.

<u>Application</u> The above Proposition can help to find the point of intersection of two tangent lines Ta and Tb to a given curve represented by a function $y = f(x)$ at two given points $A(a, f(a))$, $B(b, f(b))$ of the curve quicker that the known conventional way(s). Because (as shown in the following example after having gotten all the six necessary real numbers *a, f(a), f '(a), b, f(b), f '(b)* ready you can first solve the relation $\frac{f(b) - f(a)}{b-a} = (1-s)\,f'(a) + sf'(b)$ for s , and then consider the number $c = s\,a + (1-s)\,b$ to be the first coordinate of the point of intersection, and then find *f(c)* as for the second coordinate of the point.

<u>Example</u> Consider the cubic curve defined by $f(x) = x^3 - 3x^2 + 6$, and the two points *A(3,6)* and *B(4,22)* on the curve.

Here, $a=3$, $f(a) = 6$, $f'(a) = 9$, $b=4$, $f(b) = 22$, and $f'(b) = 24$. Then the relation $\frac{f(b) - f(a)}{b-a} = (1-s)\,f'(a) + sf'(b)$ simply becomes $16 = 9(1- s) + 24\,s$ from which we conclude $s = 7/15$. Therefore the abscissa of the point of intersection of the tangent lines to the curve is $c = s\,a + (1-s)\,b = 21/15 + 32/15 = 53/15$.

Equating slope of the line segment *AC* to $f'(a) = 9$ (the slope of *Ta*), and solving the equation for $f(c)$ we get $f(c) = 54/5$, and (*53/15*, *54/5*) would be the point of intersection.