Binomial Expansion for Negative Powers

Ali Astaneh PhD(Lon), Vancouver BC

I first recall the popular binomial expansion for positive integer powers as,

$$(a+b)^{n} = a^n + \binom{n}{1} a^{n-1}b + \binom{n}{2} a^{n-2}b^2 + \cdots + \binom{n}{n} b^n$$

Where $\binom{n}{j} = \frac{n!}{j!\,(n-j)!}$ denotes combinations of n objects taken j at a time. Or in sigma notation,

$$(a+b)^{n} = \sum_{j=0}^{n} \binom{n}{j} a^{n-j} b^{j} \quad .$$

Next, I bring the particular case of binomial expansion with negative powers as a lemma.

Lemma For any positive integer n, and any real number $-1 < x < 1$,

$$(1+x)^{-n} = 1 - \binom{n}{1} x + \binom{n+1}{2} x^2 - \binom{n+2}{3} x^3 + \cdots$$

In sigma notation this means,

$$\frac{1}{(1+x)^n} = \sum_{j=0}^{\infty} (-1)^j \binom{n+j-1}{j} x^j \qquad (1)$$

Proof: I will present the proof by mathematical induction on n.

For $n = 1$ the above identity for $-1 < x < 1$ is nothing but

$$\frac{1}{(1+x)} = 1 - x + x^2 - x^3 + \ \ldots$$

which is a known fact, say from BC Pre-Calculus 11 curriculum, as the common ratio $-x$ of the infinite geometric series $1 - x + x^2 - x^3 + \ \ldots$ is a real number strictly between -1 and 1.

In order to complete the proof, it is enough to assume that the above (1) is true and show that we also have

$$\frac{1}{(1+x)^{n+1}} = \sum_{j=0}^{\infty} (-1)^j \binom{n+j}{j} x^j \qquad (2)$$

To this end, I first differentiate both sides of the assumed identity (1) and get,

$$\frac{-n}{(1+x)^{n+1}} = \sum_{j=1}^{\infty}(-1)^j * j \binom{n+j-1}{j} x^{j-1}$$

$$= - \binom{n}{1} + \sum_{j=2}^{\infty}(-1)^j j \binom{n+j-1}{j} x^{j-1} \qquad (3)$$

Dividing both sides of (3) by - n we get,

$$\frac{1}{(1+x)^{n+1}} = 1 + \sum_{j=2}^{\infty}(-1)^{j+1} \frac{j}{n} \binom{n+j-1}{j} x^{j-1}$$

I will now make the substitution $j = k + 1$ for the index of above sigma and the indices j involved in its general term at the same time, and get

$$\frac{1}{(1+x)^{n+1}} = 1 + \sum_{k+1=2}^{\infty} (-1)^{k+2} \frac{k+1}{n} \binom{n+k}{k+1} x^k =$$

$$1 + \sum_{k=1}^{\infty}(-1)^k \frac{(n+k)!}{k!\,n!} x^k =$$

$$1 + \sum_{k=1}^{\infty}(-1)^k \frac{k+1}{n} \frac{(n+k)!}{(k+1)!\,(n-1)!} x^k =$$

$$\sum_{k=0}^{\infty}(-1)^k \binom{n+k}{k} x^k \ .$$

This completes the proof.

<u>Remark</u> : Since for $-1 < x < 1$ we have $\frac{1}{(1+x)} = 1 - x + x^2 - x^3 + \ \dots$, the above Lemma also implies the rare occurrence of a concrete example where the n^{th} power of a convergence infinite series could be another convergence series, that is, for $-1 < x < 1$,

$[1 - x + x^2 - x^3 + \ \dots]^n = 1 - \binom{n}{1} x + \binom{n+1}{2} x^2 - \binom{n+2}{3} x^3 + \cdots$.

Or, in sigma notation

$$\left[\sum_{k=0}^{\infty}(-1)^k \ x^k\right]^n = \sum_{k=0}^{\infty}(-1)^k \binom{n+k}{k} x^k \ .$$

I now bring the generalization of the above Lemma as a proposition.

Proposition : Let $a, b \neq 0$ be real numbers and let $-\left|\frac{b}{a}\right| < x < \left|\frac{b}{a}\right|$.
Then for any positive integer n we have the following expansion,

$$(ax+b)^{-n} = \frac{1}{b^n} - \frac{a}{b^{n+1}}\binom{n}{1}x + \frac{a^2}{b^{n+2}}\binom{n+1}{2}x^2 - \frac{a^3}{b^{n+3}}\binom{n+2}{3}x^3 + \cdots$$

In sigma notation this means,

$$\frac{1}{(ax+b)^n} = \sum_{j=0}^{\infty}(-1)^j \frac{a^j}{b^{n+j}} \binom{n+j-1}{j} x^j .$$

Proof: Since $\frac{1}{(ax+b)^n} = \frac{1}{b^n}\frac{1}{(1+(\frac{ax}{b}))^n}$ and since $-\left|\frac{b}{a}\right| < x < \left|\frac{b}{a}\right|$ implies $-1 < ax/b < 1$, the above Lemma (when x is replaced by (ax/b)) leads to

$$\frac{1}{(ax+b)^n} = \frac{1}{b^n}\frac{1}{\left(1+\left(\frac{ax}{b}\right)\right)^n} = \frac{1}{b^n}[\sum_{j=0}^{\infty}(-1)^j \binom{n+j-1}{j}\left(\frac{ax}{b}\right)^j] = \sum_{j=0}^{\infty}(-1)^j \frac{a^j}{b^{n+j}}\binom{n+j-1}{j} x^j$$

and the proof is complete.

Corollary: Binomial Expansion for Negative Powers

Let $0 < |a| < |b|$, then if you set $x = 1$ in the assertion of the above Proposition, you get the following binomial expansion for negative powers described as :

$$(a+b)^{-n} = \frac{1}{b^n} - \binom{n}{1}\frac{a}{b^{n+1}} + \binom{n+1}{2}\frac{a^2}{b^{n+2}} - \binom{n+2}{3}\frac{a^3}{b^{n+3}} + \cdots \quad .$$

Or,

$$(a+b)^{-n} = \sum_{j=0}^{\infty}(-1)^j \binom{n+j-1}{j} \frac{a^j}{b^{n+j}}$$