n Times "Smooth" Interpolation between an Horizontal to an Oblique Line

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To fix ideas, and in analytical geometric terms, given an horizontal half line defined by, say $y = 0,\ x \leq 0$ and any oblique half line passing through the point $(1,1)$ with equation $y - 1 = p(x - 1),$ $1 \leq x$, (or $l(x) = p(x - 1) + 1,\ 1 \leq x$), a Calculus question arises whether there exists an interpolating polynomial function whose graph connects the two end points $(0,0)$ and $(1,1)$ of the half lines in a sufficiently smooth manner. Technically speaking the question is whether there exists a polynomial function $y = P(x)$ such that the piecewise function defined by

$$f_n(x) = \begin{cases} 0 & x \leq 0 \\ P(x) & 0 < x < 1 \\ l(x) & 1 \leq x \end{cases}$$

is an n times entirely differentiable function over the real number line?

The answer to this question is affirmative, and here goes the answer,

Proposition: For any positive integer n, the function

$$f_n(x) = \begin{cases} 0 & 0 \leq x \\ x^{n+1}\left[1 + \sum_{j=1}^{n} (-1)^j \left[\binom{n+j}{j} - (2j-1)\right](x-1)^j \right] & 0 < x < 1 \\ 1 & 1 \leq x \end{cases}$$

is an entirely n times differentiable function piecewise function, where $\binom{n+j}{j}$ means the familiar number representing combination of $n + j$ objects taken j at a time.

This means the curve segment define by the polynomial in the middle of the piecewise function defined above provides a curve segment to patch up the horizontal line half line $y = 0$, $x \leq 0$ and the half slant line (through the point $(1,1)$)) defined by $y = px + 1 - p, x \geq 1$ patches up the half lines in an n times smooth manner at the two contact points.

<u>Proof:</u> I only need to remark that, obviously for each $n = 1,2,3,\ldots$ the graph of the above entirely defined function $f_n(x)$ does go/ cross through the end points of the indicated half lines, that is :

$f_n(x) = 0$ for $x \le 0$ and $f_n(x) = px + 1 - p$ for $x \ge 1$.

Otherwise, the proof is accomplished by induction on ; and it is left as an exercise to the interested reader! I should like to point out that in the process of an inductive proof, a familiarity about identities among combinations ${}_{n+j}C_j$ for distinct positive integers $n + j$ and j is required.

The following 4 concrete Examples, can give a grasp of the analytic geometrical aspects of the discussion. In all examples the left hand half line is the negative part of the x-axis, but the half slant line on the right are different. Until I find time to graphs of Examples 2-4 in my ideal satisfaction, I am asking the reader in Examples 2-4 to ignore the part of the straight line $l(x) = p(x-1) + 1$ to the left of the point $(1,1)$, as well as the polynomial graph $y = P(x)$ which is to the right of the same point $(1,1)$.

<u>Example 1</u>, For $n = 1$ and $p = -1$, the function $f_1(x)$ is,

$$f_1(x) = \begin{cases} 0 & x \le 0 \\ x^2(4 - 3x) & 0 < x < 1 \\ 2 - x & 1 \le x \end{cases}$$

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> g := x → piecewise(x < 0, 0, x < 1, 4x^2 − 3x^3, x < 3, 2 − x);
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$$g := x \to piecewise\left(x < 0, 0, x < 1, 4x^2 - 3x^3, x < 3, 2 - x\right)$$

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> plot(g(x), x = -1..2.5);
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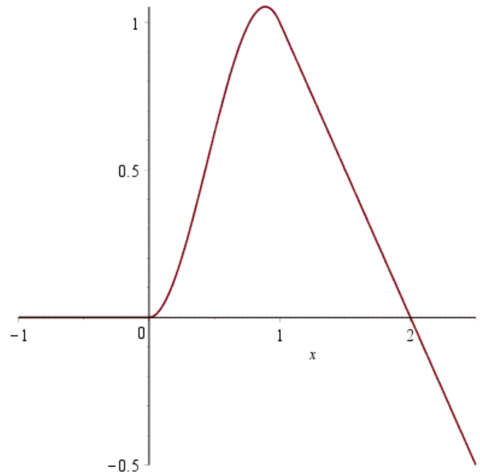


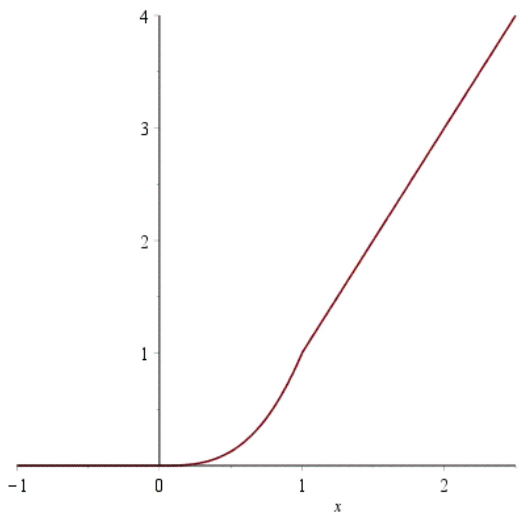
Example 2, For $n = 2$ and $p = 2$, the function $f_2(x)$ is,

$$f_2(x) = \begin{cases} 0 & x \le 0 \\ x^3(6x^2 - 15x + 10) & 0 < x < 1 \\ 2x - 1 & 1 \le x \end{cases}$$

> $h := x \rightarrow piecewise(x < 0, 0, x < 1, x^3(6x^2 - 15x + 10), x < 3, 2x - 1);$

$$h := x \rightarrow piecewise(x < 0, 0, x < 1, x(6x^2 - 15x + 10)^3, x < 3, 2x - 1)$$

> $plot(h(x), x = -1..2.5);$



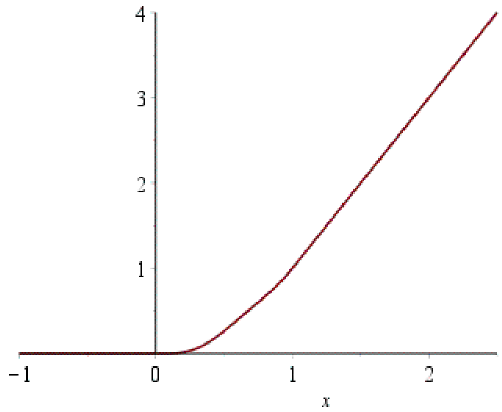
Example 3, For $n = 3$ and $p = 2$, the function $f_3(x)$ is,

$$f_3(x) = \begin{cases} 0 & x \le 0 \\ x^4(-10x^3 + 34x^2 - 40x + 17) & 0 < x < 1 \\ 2x - 1 & 1 \le x \end{cases}$$

> $k := x \rightarrow piecewise(x < 0, 0, x < 1, 17x^4 - 40x^5 + 34x^6 - 10x^7, x < 3, 2x - 1);$

$$k := x \rightarrow piecewise(x < 0, 0, x < 1, 17x^4 - 40x^5 + 34x^6 - 10x^7, x < 3, 2x - 1)$$

> $plot(k(x), x = -1..2.5);$



Example 4, For $n = 4$ and $p = 5$, the function $f_4(x)$ is,

$$f_4(x) = \begin{cases} 0 & x \leq 0 \\ x^5(10x^3 - 30x^2 + 30x + 1) & 0 < x < 1 \\ 5x - 4 & 1 \leq x \end{cases}$$

> $w := x \to piecewise(x < 0, 0, x < 1, x^5(10\,x^3 - 30\,x^2 + 34\,x + 1), x < 3, 5\,x - 4);$

$$w := x \to piecewise\left(x < 0, 0, x < 1, x\left(10\,x^3 - 30\,x^2 + 34\,x + 1\right)^5, x < 3, 5\,x - 4\right)$$

> $plot(w(x), x = -1\,..2.5);$