Poetic Characterizations for Parabolas

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I have called characterization (II) in the following theorem "Poetic" as for the existing rhyme between the words slpope and average.

Theorem (Astaneh) Let $y = f(x)$ be a nonlinear continuously differentiable function. Then the following conditions are equivalent.

(I) $f(x)$ is a quadratic function.

(II) For any two points $A(a, f(a))$ and $B(b, f(b))$, $a < b$,the graph of $f(x)$, the **average** between ***slopes*** of the tangent lines to the graph of f at the points A and B is the same as the ***slope*** of the tangent line to the graph of f at the **average** abscissa $x = (a+b)/2$ between $x = a$ and $x = b$. That is, if for any two real numbers a and b we have,

$$[f'(a) + f'(b)]/2 = f'[(a+b)/2]. \qquad (1)$$

(III) For any point $M(u, f(u))$ on the graph of $f(x)$ and any real number p, slope of the tangent line at the point $M(u, f(u))$ is the same as the slope of the secant line segment with the end points $A(u-p, f(u-p))$ and $B(u+p, f(u+p))$ on the graph of f. That is ,

$$f'(u) = [f(u+p) - f(u-p)]/2p. \qquad (2)$$

Proof: The proof that (I) implies (II) is straightforward, and is best to be considered as an exercise for Calculus students. To show (II) implies (III), we first observe that upon substitution $u = (a+b)/2$ and $u = (b-a)/2$ the relation (1) can be expressed as

$$f'(u+p) + f'(u-p) = 2f'(u). \qquad (3)$$

Next, keeping u fixed, and integrating both sides of (3) with respect to p implies

$$f(u+p) + f(u-p) = 2f'(u)p + C \ . \qquad (4)$$

Taking limits from both sides of relation (4) as p approaches zero implies $C = 0$, and therefore The relation (2), as

$$f'(u) = [f(u+p) - f(u-p)]/2p,$$

is established.

To prove (III) implies (I) I will make use of my previous characterization for parabolas seen as Part (II) of the Theorem on page 16, issue #15 of 2011 edition of the "Pi in the Sky" magazine (which happens to be the tenth item in this Calculus 1 section of this same Website) . According to characterization presented there, it is enough to show that for any two real numbers a and b we have

$$[f'(a)+f'(b)]/2=[f(b)-f(a)]/(b-a)\,. \qquad (5)$$

To this end, let a and b be arbitrary and set choose $u=(a+b)/2$ and $p=(b-a)/2$, then for these choices of u and p the assumed relation (2) also holds, that is,

$$f'(u)=[f(u+p)-f(u-p)]/2p\,,$$

or,
$$f(u+p)-f(u-p)=2p\,f'(u)\,.$$

Differentiating both sides of the above relation with respect to p we get,

$$f'(u+p)+f'(u-p)=2\,f'(u)\,.$$

Again, using (2) for the right hand side the above relation becomes

$$f'(u+p)+f'(u-p)=\left[f(u+p)-f(u-p)\right]/p\,.$$

And finally, substituting back $u=(a+b)/2$ and $p=(b-a)/2$ we get

$$f'(b)+f'(a)=2\left[f(b)-f(a)\right]/(b-a)2\,,$$

From which (5) is established, and the proof is complete.