A Direct Formula For the n^{th} Derivative of a Function

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Sometime around November 2012, of my own curiosity, I had become interested to know if there is a direct formulation to express the $n\,th$ derivative $f^{(n)}(x)$ of an n times differentiable function directly as a single formula, just as we express the first derivative as a limit of a fraction in the "variable" h. It took me a reasonably long while to figure out that the answer to my question was affirmative, and I finally came up with the following formulation which I formally record here as a theorem.

Theorem Let $f(x)$ be an n times continuously differentiable function (that is we assume that $f^{(n)}(x)$ is continuous). Then for any real number x we have,

$$f^{(n)}(x) = lim_{h\to 0} h^{-n} \sum_{j=0}^{n} (-1)^j \binom{n}{j} f(x + [n-j]h). \quad (1)$$

Here, as usual $\binom{n}{j}$ means the combinations of n objects taken k at a time.

Proof It is enough to show that the right hand limit in (1) is the same as $f^{(n)}(x)$.
To this end, it is just the matter of using L'Hopital's rule n times to the limit on the right. Then you end up with the limit as h. Goes to zero of the limit

$$\frac{1}{n!}\{ [n^n \binom{n}{0} - (n-1)^n \binom{n}{1} + (n-2)^n \binom{n}{2} + \ \dots \ + (-1)^{(n-1)} \binom{n}{n-1}] \} f^n(x+nh).$$

Since t $f^{(n)}(x)$ is assumed to be continuous, the above limit is

$$\frac{1}{n!}\{ [n^n \binom{n}{0} - (n-1)^n \binom{n}{1} + (n-2)^n \binom{n}{2} + \ \dots \ + (-1)^{(n-1)} \binom{n}{n-1}] \} f^n(x).$$

Now I quote the remarkable identity for $n!$ discussed in the Probability and Combinatorics section of this same website, which goes like,

$$n^n \binom{n}{0} - (n-1)^n \binom{n}{1} + (n-2)^n \binom{n}{2} + \ \dots \ +(-1)^{(n-1)} \binom{n}{n-1} = n!$$

This implies that the expression inside curly bracket { … } in latter expression above is just equals to $n!$, and therefore the proof of the theorem is complete.

For further justification; and actually going beyond justification , here is an example for a specific function to see how my limit in (1) works.

Example: Use limit (1) in the Theorem to find $f''(x)$ for $f(x) = \sin x$.

$$f''(x) = \lim_{h\to 0} [\ \sin(x+2h) - 2\sin(x+h) + \sin(x)] / h^2 \ =$$

$$f''(x) = \lim_{h\to 0} [\ \sin(x+2h) + \sin(x) - 2\sin(x+h)] / h^2 \ =$$

$$f''(x) = \lim_{h\to 0} [\ 2\sin(x+h)\cos(h) - 2\sin(x+h)] / h^2 \ =$$

$$f''(x) = 2\lim_{h\to 0} \sin(x+h)[\cos(h) - 1] / h^2 \ =$$

$$f''(x) = 2\sin(x)\ \lim_{h\to 0}[\cos(h) - 1] / h^2 \ =$$

$$f''(x) = -2\sin(x)\ \lim_{h\to 0}[1 - \cos(h)] / h^2 = -\sin(x)\ .$$