General Inverse Functions, The "Slant Line" Test

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If you consider the point of view that in analytic geometry two functions $f(x)$ and $g(x)$ are regarded to be _inverse_ of each other when their graphs are mirror images of each other through the line $y = x$, the intuition persuades one to call two functions $F(x)$ and $G(x)$ the _general inverse_ of each other if instead their graphs are the mirror images of each other through a _general_ line such as $y = mx + b$. The intuition then quickly raises the question about the nature and the validity of an identity between functions $F(x)$ and $G(x)$ matching that of $f(g(x)) = x$, known for the case when the mirror line is $y = x$.

Like any other problem in math, you would find it easier to break down the question/problem into simpler cases first, and so I did. More precisely, in a string of postings from late March to mid October 2011, I showed (as the proofs below will reveal shortly) on the BCAMT public list-serve that:

(I) If the mirror line is $L : y = x + b$, then the matching identity is:

$$F(G(x - b) - b) = x .$$

I originally called functions satisfying the above particular identity _b- partial inverse_ of each other.

Example 1 Graphs of the two functions $F(x) = (x - 2)^3 + 3$ and $G(x) = \sqrt[3]{x} + 5$ are mirror images of each other through the line $L : y = x + 3$. Since here $b = 3$, the following identity must hold between $F(x)$ and $G(x)$:

$$F(G(x - 3) - 3) = x.$$

A direct verification of this identity should be a good exercise for a Grade 11 Math student.

(II) If the mirror line is $L' : y = mx$, and if $\angle\theta$ represents the angle of elevation

of the line L' from the positive $x - axis$ (that is, if $m = \tan\theta$) then the matching identity is

$$F((\sin2\theta)G(x) + (\cos2\theta)x) = -(\cos2\theta)G(x) + (\sin2\theta)x .$$

I originally called functions satisfying either of the above (equivalent) identities

<u>*m- skew inverse functions*</u> .

 Mind you, in case you are rather an algebra person, obsessed to have your answer only in terms of the slope m of the line L'; then you will have to deal with the following less handsome looking identity:

$$F([2mG(x)+(1-m^2)x]/(1+m^2)\ ) = [\ (m^2-1)G(x)+2mx]\ /\ (1+m^2\ ).$$

Here, the bridge between the above two versions being the following trigonometric identities:

$$\sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta} = \frac{2m}{1+m^2}\ ,\qquad \cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta} = \frac{1-m^2}{1+m^2}\ ,$$

where, noticeably, $m = \tan\theta$.

<u>Example 2</u> Consider the mirror line to be L': $y = 2x$, where $m = 2$. For simplicity of verifications, I will bring an example where $F(x)$ and $G(x)$ are linear functions. First, one can verify that graphs of the two linear functions $F(x) = \frac{1}{2}x - 3$ and $G(x) = -\frac{11}{2}x - 15$ are reflections of each other through the line L', noting that the three lines involved meet at the point $(-2,4)$. Therefore, considering that $\tan\theta = 2$ implies $\sin 2\theta = \frac{4}{5}$ and $\cos 2\theta = -\frac{3}{5}$, the two functions $F(x)$ and $G(x)$ satisfy the following identity:

$$F(\frac{4}{5}G(x) - \frac{3}{5}x\ ) = \frac{3}{5}G(x) + \frac{4}{5}x\ ;$$

direct verifications of which should be another good exercise for a Grade 11 Math student.

(III) If the mirror line is L'': $y = mx + b$, and again if $\angle\theta$ represents the angle of elevation of the line L'' from the positive $x - axis$, the matching identity is:

$$F(\,(\sin2\theta)[G(x)-b]+(\cos2\theta)x\,)=\,-(\cos2\theta)[G(x)-b]+(\sin2\theta)x\,+b\,.$$

I originally called two functions satisfying the above identity to be the _general inverse_ of each other with respect to the line L'': $y=mx+b$.

Again, if you would like to avoid trigonometry, then your identiy in terms of the slope m, and the $y-intercept\ b$ of the line L'' would be the less good looking identity:

$$F([2m(G(x)-b)+(1-m^2)x]/(1+m^2))=[\,(m^2-1)G(x)+2(mx+b)]\,/\,(1+m^2\,)$$

Example 3 Consider the mirror line to be L'': $y=2x+3$, where $m=2$ and $b=3$ .The two linear functions $F(x)=\dfrac{1}{2}x$ and $G(x)=-\dfrac{11}{2}x-12$ are reflections of each other through the line L'', noting that the three lines meet at the point $(-2,-1)$. Therefore the two functions $F(x)$ and $G(x)$ must satisfy the following identity,

$$F(\frac{4}{5}[G(x)-3]-\frac{3}{5}x\,)=\,\frac{3}{5}[G(x)-3]+\frac{4}{5}x+3\,.$$

Perhaps this time, the direct verification of the above relation would be a more challenging exercise for a Grade 11 Math student.

I trust the reader of the article will easily be able to conclude the validity of the algebraic version of the above assertion **_(III)_** (and therefore in particular of **_(I)_** and **_(II)_**) from the following _Proposition_ that I also presented in another context about 8-10 years ago in a Vancouver School District annual mini-conference. Once you acknowledge the validity of the algebraic version of **_(III)_**, to verify its trigonometric version is only a matter of substituting from the following well-known identities:

$$\sin2\theta=\frac{2\tan\theta}{1+\tan^2\theta}=\frac{2m}{1+m^2}\,,\qquad\cos2\theta=\frac{1-\tan^2\theta}{1+\tan^2\theta}=\frac{1-m^2}{1+m^2}\,.$$

I first bring the following routine _Lemma_, to smooth the way for proof of the _Proposition._

Lemma: Given a point $P\ (x,y)$ and a line $L: y = mx + b$ on the coordinate plane, the reflection $Q\ (X,Y)$ of the point P through the line L has coordinates

$$Q\ (X,Y) = Q\ (\frac{(1-m^2)x + 2m(y-b)}{1+m^2}, \frac{(m^2-1)y + 2(mx+b)}{1+m^2}).$$

If the line is vertical, that is, with an equation such as $L: x = k$, then the reflection Q of the point $P\ (x,y)$ through L will have coordinates $Q\ (2k - x, y)$.

y = mx + b

P(x,y)

H(u,mu+b)

Q(X,Y)
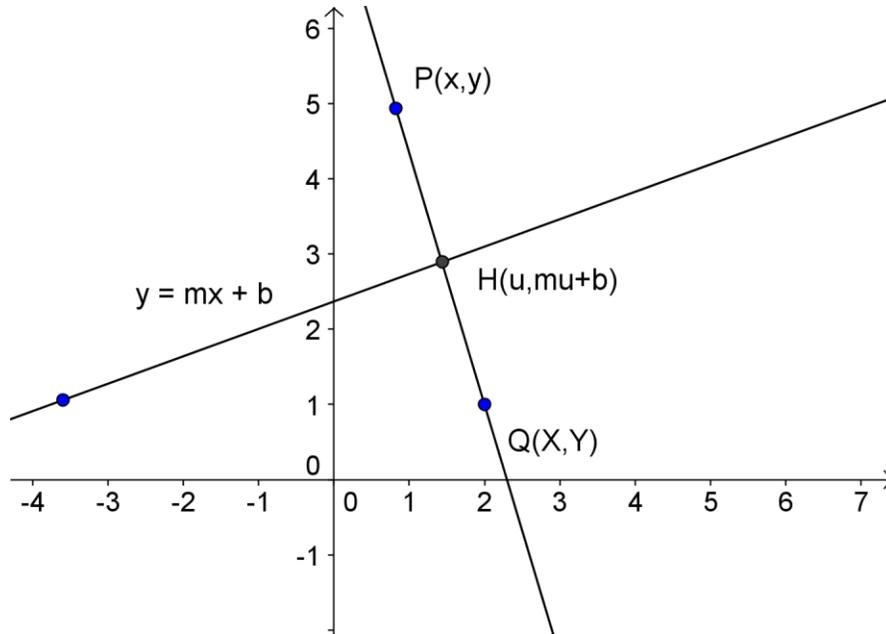
Proof : Let $Q\ (X,Y)$ be the reflection of $P\ (x,y)$ through the line L, and let $H(u, mu+b)$ be the point of intersection of L and line segment PQ. Then, as L and PQ are perpendicular we have

$\dfrac{mu+b-y}{u-x} = -\dfrac{1}{m}$, and therefore $u = \dfrac{x+m(y-b)}{1+m^2}$. It then follows that H has coordinates

$$H(\frac{x+m(y-b)}{m^2+1}, \frac{m(x+my)+b}{m^2+1})\ .$$

Since H is the midpoint between of P and Q, one can use the midpoint formula to show that

$$X = \frac{2m(y-b)+(1-m^2)x}{1+m^2}, \text{ and } Y = \frac{(m^2-1)y + 2(mx+b)}{1+m^2}\ .$$

Later in the article we will study conditions under which mirror image of the graph of a given function $Y = F(X)$ through a line $L: y = mx + b$ will represent a

"function" as opposed to a "relation". However, since in assertion of *(III)* above it is assumed that the mirror image does represent a "function", we will make such an assumption in the following *Proposition*.

Proposition: Assume that the mirror image of a function $Y = F(X)$ through a line $L : y = mx + b$ is the graph of a function $y = G(x)$. Then for every x in $Dom(G)$ the following relation holds

$$F([2m(G(x) - b) + (1 - m^2)x]/(1 + m^2)) = [(m^2 - 1)G(x) + 2(mx + b)]/(1 + m^2)$$

Proof : Since, when a point $P(x, y) = P(x, G(x))$ is on the graph of $G(x)$, by above *Lemma* the reflection transformation of this point is

$$(x, G(x)) \to (\frac{2m(G(x) - b) + (1 - m^2)x}{1 + m^2}, \frac{(m^2 - 1)G(x) + 2(mx + b)}{1 + m^2}),$$

and since the ordered pair on the right hand side has to be on the graph of $Y = F(X)$, we must have

$$F(\frac{2m(G(x) - b) + (1 - m^2)x}{1 + m^2}) = \frac{(m^2 - 1)G(x) + 2(mx + b)}{1 + m^2},$$

from which the *Proposition* follows.

Remark: I trust the reader of the article realizes that each one of the identities in the above *(I)-(III)* assertions has *a dual counterpart* identity which is simply obtained by switching the letters F and G around. This is all because, the same relations that express X and Y in terms of x and y will likewise express of x and y in terms of X and Y.

Also, note that the *mapping*

$$(x, y) \to (\frac{2m(y - b) + (1 - m^2)x}{1 + m^2}, \frac{(m^2 - 1)y + 2(mx + b)}{1 + m^2})$$

obtained from the *Lemma* can be used to find equation of the reflection curve of any pre-assigned original curve defined by a given relation, as seen in the following example.

Example 4 Consider the mirror line to be $L : y = 2x - 3$, where $m = 2$ and $b = -3$. The above *mapping* implies that equation of the reflection of the hyperbola $X^2 - 4Y^2 - 4 = 0$ through the line $L : y = 2x - 3$ will be another hyperbola with

$$(\frac{-3x+4y+12}{5})^2 - 4(\frac{4x-6+3y}{5})^2 - 4 = 0.$$

Obviously the asymptotes of this hyperbola are no longer horizontal nor vertical (like it was for the original hyperbola), however one can obtain the equations of those asymptotes by finding the transformations of the original asymptotes under the above mapping, as $x \pm 2y = 0$.

I continue the article by presenting two theorems describing two different ways of presenting the necessary and sufficient conditions under which the general inverse of a given function $F(x)$ through a line $L : y = mx + b$ is a *"function"*, as opposed to being a "*relation*".

I naturally have called the first theorem the "slant line test" test, because of the slant line involved in part (II). I also have called the second theorem the" rotated horizontal test " because of the idea that the horizontal test is used to the counterclockwise rotation of the graph of the original function by an angle of $\frac{\pi}{2} - 2\theta$ radians, where $\angle\theta = \tan^{-1} m$ represents the angle of elevation of the line L' from the positive $x - axis$.

Note that, in the light of the above *Remark* (and for convenience in recording) we can denote the argument of an origin function F to be small character x rather than X . That is, we will simply write $y = F(x)$ in what follows.

Theorem 1 "slant line test": Given a function $y = F(x)$, and a line $L : y = mx + b$ on the coordinate plane,

(I) If the <u>range</u> of the function $y = F(x)$ is known to be the closed interval $[c, d]$, then the <u>domain</u> of the mirror image curve will be the following related closed interval:

$$[A, B] = [\frac{(1 - m^2)}{(1 + m^2)} + mc + b , \ \frac{(1 - m^2)}{(1 + m^2)} + md + b].$$

(I)The mirror image of the graph of $y = F(x)$ through the line $L : y = mx + b$ will represent a "*function*" if and only if for every number k in the interval $[c,d]$ the "slant" line defined by

$$SL: y = [(m^2 - 1) / 2m] \, x + k$$

intersects the graph of the original function $y = F(x)$ at most at one point.

Note that, here two remarks are in order; first if instead the range of the original function $y = F(x)$ happens to be the entire real number line, so will be the domain of the mirror image, which then means k should be considered to to be any real number. Secondly, as expected, when $m = 1$ and $b = 0$, (that is, when the mirror line is $L : y = x$) it can easily be seen that the above test is reduced to the familiar "horizontal line test" for deciding whether the usual inverse of the function $y = F(x)$ is a "function".

Proof of the Theorem:

proof of part (I) Let the range of the function $y = F(x)$ be the closed interval $[c,d]$. Then the inverse transformation

$$(x, y) \rightarrow (\; \frac{2m(y-b)+(1-m^2)x}{1+m^2} \; , \; \frac{(m^2-1)y+2(mx+b)}{1+m^2} \;)$$

mentioned in the remark prior to Example 4 implies that the respective mirror images of the points $(0,c)$ and $(0,d)$ are

$$(0,c) \rightarrow (\frac{2m(c-b)}{1+m^2}, \frac{(m^2-1)c+2b}{1+m^2})$$

$$(0,d) \rightarrow (\frac{2m(d-b)}{1+m^2}, \frac{(m^2-1)d+2b}{1+m^2}).$$

Therefore, as claimed, the domain of inverse function $G(x)$ must be the closed interval

$$[A,B] = [\frac{2m(c-b)}{1+m^2}, \frac{2m(d-b)}{1+m^2}].$$

Proof of part (II) Since, by the usual vertical line lest graph of $G(x)$ will represent a "function" if and only if any vertical line $x = k$, with k in the interval

$[A,B]$, intersects $G(x)$ at most at one point, it follows that this will be the case if and only if the mirror image of the line $x = k$ through the mirror line

$L: y = mx + b$ (which would now be a <u>slant line</u> unles $\theta = \tan^{-1} m = \pi / 4$ or

$\theta = \tan^{-1} m = 3\pi / 4$) intersects the graph of the original function $F(x)$ at most at one point. On the other hand since the mirror images of the points (k, y) on the vertical line $x = k$ are the points with given coordinates

$$(k,y) \to (\frac{2m(y-b) + (1-m^2)k}{1+m^2}, \frac{(m^2-1)y + 2(mk+b)}{1+m^2}),$$

to find the equation of the mirror image $LS : Y = MX + B$ of the "slant line", we only need to eliminate the parameter y from in the system of equations

$$X = \frac{2m(y-b) + (1-m^2)k}{1+m^2}$$

$$Y = \frac{(m^2-1)y + 2(mk+b)}{1+m^2}.$$

The reader can verify that eliminating y from the above system will result in the following relation

$$SL: y = \frac{(m^2-1)}{2m}x + \frac{(m^2+1)}{2m}bk$$

as the equation of the mirror image of the vertical line $x = k$ through the mirror line $L: y = mx + b$. Hence the proof of **part (II)** is also complete.

In order to see how the slant line test works in action, the following example is in order.

Example 5 Let the mirror line be $L: y = 2x - \dfrac{2}{5}$, and consider the original curve

segment to be the graph of the function $F(x) = x^2 / 4$ <u>restricted</u> to the closed

Interval $[-1,4]$, and therefore $F(x)$ having the range $[c,d] = [\dfrac{1}{4}, 4]$. To perform

the "slant line test" as an *algorithmic* proceed as in the following steps:

Step 1. Look at the line $L: y = 2x - \dfrac{2}{5}$ as $L: y = mx + b$ and then, for any real

number k in the interval

$$[A,B]=[\frac{2m(c-b)}{1+m^2},\frac{2m(d-b)}{1+m^2}]=[\frac{13}{25},\frac{98}{25}]$$

form equation of the *"slant"* line defined by

$$SL: y = \frac{(m^2-1)}{2m}x+\frac{(m^2+1)}{2m}bk = \frac{3}{4}x-\frac{1}{2}k \ .$$

Step 2. Investigate whether any such a "*slant*" line $SL: y = \frac{3}{4}x-\frac{1}{2}k$ can

possibly intersect graph of the function $F(x) = x^2/4$ at more than one point

when $F(x) = x^2/4$, for any k in the interval $[A,B]=[\frac{13}{25},\frac{98}{25}]$, which means

determine whether the equation

$\frac{x^2}{4}=\frac{3}{4}x-\frac{1}{2}k$ for any k in the interval $[\frac{13}{25},\frac{98}{25}]$. As a routine exercise in Math

11, since the "discriminant" of the quadratic equation $x^2-3x-2k=0$ is

$9+8k$, and since for any k in the interval $[\frac{13}{25},\frac{98}{25}]$ the "discreminant" is

positive the "slant line test" Theorems fails the grapgh of the general function , as representing a "function"

Exercise. As a related exercise to the above example, find a unique rational

number u such that the mirror image of the graph $F(x) = x^2/4$ through

$L: y = 2x-\frac{2}{5}$ never will be a "function" if the domain of $F(x)$ is chosen to be

any interval containing u as its <u>inerior point</u>.

I close the article by presenting a different test to investigate whether the general inverse of a given function $F(x)$ through a line $L: y = mx+b$ is a "function", as opposed to being a "relation". Although I have taken an algebraic approach to prove the theorem (only to make it available to more gifted Math 11 students as well), a trigonometric approach would have been much shorter, considering that (as I have mentioned before) the mirror image of the graph of a function $y = F(x)$ through a line $L: y = mx+b$ can be obtained as by using

trigonometry on the angle $\angle 2\theta$, where $\angle\theta = \tan^{-1}m$ is the angle of elevation of the line L' from the positive $x-axis$.

Theorem 2 "rotated horizontal test": Given a function $y = F(x)$, and a line $L : y = mx + b$ on the coordinate plane, the reflection of the graph of $F(x)$ through the line *L* will represent a "*function*" if and only if the _auxiliary_ function *g*, with the same domain as F, defined by $g(x) = (1-m^2)x + 2mF(x)$ is a _one to one_ function. That is, if and only if for any given x_1 and x_2 in the domain of F, the equality

$$(1-m^2)x_1 + 2mF(x_1) = (1-m^2)x_1 + 2mF(x_2) \qquad (1)$$

implies $x_1 = x_2$.

Note that, when the line is horizontal, that is when $m = 0$, the above condition (1) automatically holds, which means that, as expected, the reflection of any function through any horizontal line is already a "function".
Also, again when $m = 1$, that is when we flip $F(x)$ over the line $y = x$, or any line parallel to it, condition (1) implies that the reflection will be a function if and only if the original function $F(x)$ is one to one, again a known expected fact. Condition (1) also implies the same is true if $F(x)$ is reflected through the line $y = -x$. The above theorem doesn't deal with the case that the line *L* is vertical, however one can use the second part of the lemma to show that the reflection of any function through any vertical line is also already a function.

Proof of the Theorem : Let us denote the graph of $F(x)$ and its reflection through *L* by G_F and $G_{re(F)}$ respectively. Then the previous proposition implies that the points on $G_{re(F)}$ are parameterized as follows,

$$x \longrightarrow (\ \frac{2m\,[F(x)-h]+(1-m^2)x}{1+m^2} \ , \ \frac{(m^2-1)F(x)+2(mx+h)}{1+m^2} \ ),$$

with $x \in Dom(F)$. Considering that, visibly, the numerator of the fraction on the left can be written in terms of the auxiliary function $g(x) = (1-m^2)x + 2mF(x)$, the parameterization can be expressed as,

$$x \longrightarrow (\ G(x) \ , \ \frac{(m^2-1)}{2m}G(x) + \frac{(1+m^2)^2 x + 2mh}{2m(1+m^2)} \ ) \qquad (2)$$

where $G(x) = [g(x) - 2mh]/(1+m^2)$. Note that here, we have assumed $m \neq 0$, as the case $m = 0$ will immediately follow from the simplified parameterization $x \longrightarrow (\ x \ , \ F(x) + 2h \ )$.
We now proceed to prove the assertions of the theorem. First assume that *g(x)* is a one to one function. To show that parameterization 2*) generates a function we must show that for any

$x_1, x_2 \in Dom(F)$, the equality $G(x_1) = G(x_2)$ implies

$$\frac{(m^2-1)}{2m}G(x_1)+\frac{(1+m^2)^2 x_1+2mh}{2m(1+m^2)}=\frac{(m^2-1)}{2m}G(x_2)+\frac{(1+m^2)^2 x_2+2mh}{2m(1+m^2)} \quad (3)$$

Therefore let $x_1, x_2 \in Dom(F)$ and $G(x_1) = G(x_2)$. Since one to one-to-oneness of the auxiliary function $g(x)$ implies that $G(x)$ is also one to one, the equality $G(x_1) = G(x_2)$ first implies $x_1 = x_2$. Now it is straightforward to see that the equalities $G(x_1) = G(x_2)$ and $x_1 = x_2$ together imply (3) holds and therefore the reflection of $F(x)$ is indeed a function.

To show that $g(x)$ is one to one if the reflection is a function, we must show that for any $x_1, x_2 \in Dom(F)$, the equality $g(x_1) = g(x_2)$ implies $x_1 = x_2$. To this end let $g(x_1) = g(x_2)$, the also $G(x_1) = G(x_2)$. Since the reflection is a function, (2) and $G(x_1) = G(x_2)$ imply

$$\frac{(m^2-1)}{2m}G(x_1)+\frac{(1+m^2)^2 x_1+2mh}{2m(1+m^2)}=\frac{(m^2-1)}{2m}G(x_2)+\frac{(1+m^2)^2 x_2+2mh}{2m(1+m^2)} .$$

It is now straightforward to see that this last equation together with $G(x_1) = G(x_2)$ imply $x_1 = x_2$ and the proof is complete.

Corollary The function $g(x) = (1-m^2)x + 2mF(x)$ is just a scale (by $1+m^2$) of the counterclockwise rotation of the graph of the original function by an angle of $\frac{\pi}{2} - 2\theta$ radians, where $\angle\theta = \tan^{-1}m$ represents the angle of elevation of the line L' from the positive $x-axis$. Moreover the mentioned counterclockwise rotation is the same as graph of the general inverse of $F(x)$.

Proof. This is because we can simply write

$$\frac{1}{1+m^2}g(x)=\frac{1-m^2}{1+m^2}x+\frac{2m}{1+m^2}F(x)=(\cos2\theta)x+(\sin2\theta)F(x)$$
$$=[\sin(\frac{\pi}{2}-2\theta)]x+[\cos(\frac{\pi}{2}-2\theta)]F(x)$$

Example 6

(a)   Let $L:y=\dfrac{(\sqrt{5}+1)}{2}x-1$ and consider the reflection of the function

$f(x)=\sqrt{x-2}$ through *L*. Since the *auxiliary* function

$$g(x)=(1-m^2)x+2mf(x)=(\sqrt{5}+1)[\sqrt{x-2}-\tfrac{1}{2}x]$$

isn't one to one (indeed $g(2)=g(6)$), it follows that the reflection of

$y=\sqrt{x-2}$ through the line *L* isn't a function.

The above theorem has another application. It can be used to show that certain functions are one to one, as in the following.

One can use the theorem show that for any number $\sqrt{2}+1<m$, or

$0<m<\sqrt{2}-1$, the function $g(x)=2m|x|+(1-m^2)x$ is a one to one function. To this end consider the reflection of the function $f(x)=|x|$ through the line $L:y=mx$. Then $g(x)$ is the auxiliary function for this reflection. Since, by geometric   inspection, the reflection of $f(x)=|x|$ through $L:y=mx$ will be a

function when $\sqrt{2}+1=\tan\dfrac{3\pi}{4}<m$, and also when $0<m<\tan\dfrac{\pi}{8}=\sqrt{2}-1$, the

one to one property of $g(x)=2m|x|+(1-m^2)x$ for claimed values of *m* follows from the theorem immediately.