A Second Look at Multipliers Algorithm for Solving Polynomial Equations and Factoring

By Ali Astaneh

Ali Astaneh is a mathematics teacher at Prince of Wales Secondary in Vancouver.

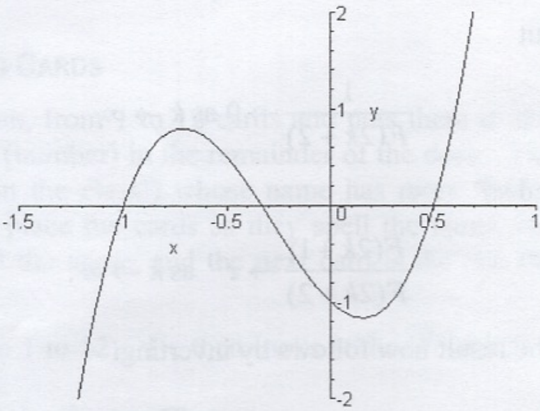
Early in 2003, while I was trying to address questions posted on the BCAMT list-serve from math teachers, I encountered a method (which I call *multipliers algorithm*) to find rational zeros of a polynomial function without using the rational zero theorem. It turned out to be very interesting. So interesting that a senior UBC math professor on the list made the following comment about it:

> *...The factoring algorithm you have described in various earlier postings is very nice, and is certainly a significant improvement on the procedures ordinarily taught...*

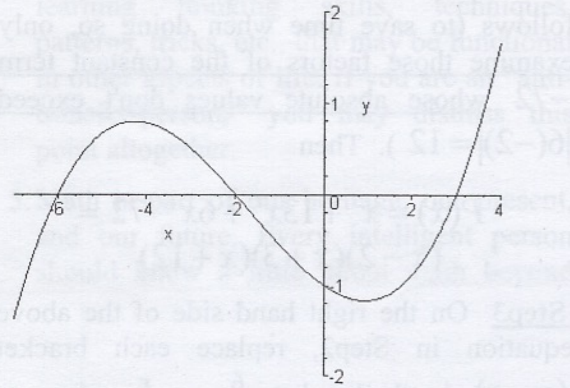
In an attempt to compensate the lack of publicity of the method and to include the many new math teachers that have come along to teach in B.C. secondary schools since 2003, I recently gave a presentation about the method in February 2009 Vancouver Math Conference at Prince of Wales Secondary for the second time; the first time being in the Fall 2003 North West Conference at Whistler. The other reason for my February presentation was to give an amazingly simpler analytical geometric account of the solution for finding rational zeros of a polynomial, before presenting the algebraic algorithm.

Since two of the math teachers attending both Whistler and Vancouver presentations told me the second time in February was much easier to understand and pleasant to follow, I thought I should revise the presentation handout and turn into a short article for the attention of interested math teachers.

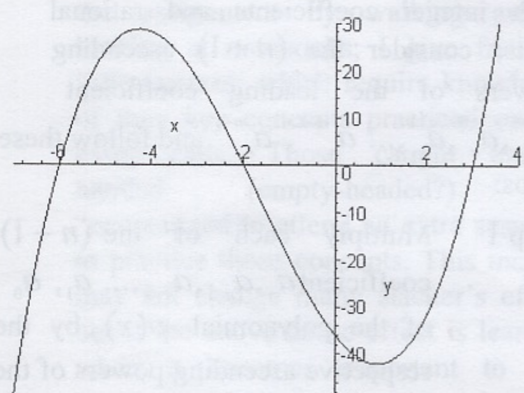
To start with, the idea behind the method is amazingly simple. Assume for the sake of argument you have a polynomial function with integer coefficients $p(x)$ which has say only three rational zeros $-1, -\frac{1}{3}$, and $\frac{1}{2}$. [Note that the software used for this article labels rational numbers on the x-axis as decimals; that is, for example, 0.5 means $\frac{1}{2}$]



If you multiply these three rational zeros by their common denominator 6 , you obtain three integers $-6, -2$, and 3 *which happen to be* the zeros of horizontal expansion of the polynomial $p(x)$ by a scale of 6 ; that is the polynomial $p(x/6)$. Except that the coefficients of this new polynomial may be non-integer rational numbers.



However, you can always vertically expand the new polynomial $p(x/6)$ by a positive integer *M*, so that the coefficients of the new polynomial $P(x) = M\,p(x/6)$ all become integers; while at the same time the integers $-6, -2$, and 3 remain the three zeros of $P(x)$. This is because a vertical expansion never changes zeros of a polynomial function. We will shortly see why that appropriate integer *M* best be chosen 6^2 ; that is [the leading coefficient of the original *p*(*x*)] raised by [degree of *p*(*x*) – 1].



Here, the advantage about working with $P(x)$ will be that you only need to use the integral zero theorem to find its integer zeros. Then you can recover the required zeros of the original polynomial $p(x)$ by dividing those integer zeros of $P(x)$ all by 6.

The following example is the precise algebraic translation of the analytical geometric approach described above.

Example 1: Let's say you would like to find rational zeros of the polynomial function

$$p(x) = 6x^3 + 5x^2 - 2x - 1.$$

If you horizontally expand this polynomial by its leading coefficient 6, as mentioned above the zeros of the new expanded polynomial $p(x/6) = \frac{1}{36}x^3 + \frac{5}{36}x^2 - \frac{1}{3}x - 1$ will be exactly 6 times zeros of $p(x)$. Since vertical expansion of a polynomial will never change its zeros, it follows that zeros of the polynomial

$$P(x) = 6^2\,p(x/6) = x^3 + 5x^2 - 12x - 36$$

will also be 6 times zeros of the original polynomial $p(x)$. Note that here the vertical scaling factor 6^2 is to be interpreted as the leading coefficient of $p(x)$ raised by the integer (degree of $p(x)$ -1) . One can now use the integral zero theorem, and factor the auxiliary polynomial $P(x)$ as follows

$$P(x) = (x-3)(x+6)(x+2).$$

Since the three zeros of $P(x)$ are $x = -6, -2, 3$ we obtain the three rational roots of the original polynomial $p(x)$ as

$$x = -\frac{6}{6}, -\frac{2}{6}, \frac{3}{6}$$

which means $x = -1, -\frac{1}{3}, \frac{1}{2}$.

Note, that rational zero theorem was put out of business here; and that is what I meant by " redundancy of rational zero theorem in dealing with polynomial equations and factoring“ in the title of my February presentation.

Since the problem of finding rational zeros of a polynomial is in the PMath11 curriculum in B.C., and at that level students haven't learned about horizontal and vertical expansion transformations of functions, I later converted the above analytical geometric approach into an algebraic algorithm, without any mention of horizontal or vertical expansion of polynomials. For the obvious reasons we shall see shortly I called the method “ The Multipliers Algorithm”. I will now bring a second example to explain how we go through the three steps of the algorithm.

Example 2: Let's say you would like to factor the polynomial

$p(x) = 6x^3 + 13x^2 + x - 2$.

Since there are four coefficients for this polynomial, consider the four ascending powers $6^{-1}, 6^0, 6^1, 6^2$ of the leading coefficient 6 of the polynomial, and follow these steps:

<u>Step1</u> Multiply each one of the four ascending powers $6^{-1}, 6^0, 6^1, 6^2$ in order by the respective coefficients (from left) of the original polynomial $6x^3 + 13x^2 + x - 2$ to obtain a new auxiliary polynomial, which means

$$P(x) = x^3 + 13x^2 + 6x - 72.$$

Remark: This <u>Step1</u> will horizontally expand the polynomial $p(x)$ by 6 and then vertically expand the result by 6^2 at the same time to obtain the auxiliary polynomial

$$P(x) = 6^2 p(x/6)$$

whose roots are 6 times those of $p(x)$ with integer coefficients.

<u>Step2</u> Use the integral zero theorem to factor the auxiliary polynomial $P(x)$ as follows (to save time when doing so, only <u>examine those factors of the constant term</u> -72 <u>whose absolute values don't exceed</u> $|6(-2)| = 12$). Then

$$P(x) = x^3 + 13x^2 + 6x - 72 = (x-2)(x+3)(x+12).$$

<u>Step3</u> On the right hand side of the above equation in Step2, replace each bracket $(x - r)$ by the $(x - \frac{r}{6})$ to get $(x - \frac{1}{3})(x + \frac{1}{2})(x - 2)$. Then $p(x) = 6(x - \frac{1}{3})(x + \frac{1}{2})(x - 2)$ is the required factorization of the polynomial $p(x) = 6x^3 + 13x^2 + x - 2$.

The Multipliers Algorithm in General

To factor the polynomial

$p(x) = a_n x^n + a_{n-1}x^{n-1} + \ ... \ + a_1 x + a_0$

with integer coefficients and rational roots, consider the $(n+1)$ ascending powers of the leading coefficient $a_n^{-1}, a_n^0, a_n^1, ... \ a_n^{n-2}, \ a_n^{n-1}$ and follow these steps:

Step 1 Multiply each of the $(n+1)$ coefficient $a_n, a_{n-1}, a_{n-2}, ... \ a_1, \ a_0$ of the polynomial $p(x)$ by the respective ascending powers of the leading coefficient a_n, that is by $a_n^{-1}, a_n^0, a_n^1, ... \ a_n^{n-2}, \ a_n^{n-1}$ to obtain a new auxiliary polynomial

$$P(x) = x^n + c_{n-1}x^{n-1} + \ ... \ + c_1 x + c_0$$

Step 2 Use the **integral zero theorem** to factor the auxiliary polynomial $P(x)$, say as follows (when doing so only test only those factors of the constant term

$$c_0 = a_n^{n-1} a_0$$

whose absolute values don't exceed $|a_n a_0|$). (*)

$$x^n + c_{n-1}x^{n-1} + \dots + c_1 x + c_0 = (x-r_1)(x-r_2)\dots(x-r_n)$$

Here $r_1, r_2, \dots, r_n$ denote all integer zeros of $P(x)$.

Step 3 For $k = 1, 2, \dots, n$ let r_k denote the k^{th} integer root of $P(x)$, obtained by Integral zero theorem for $P(x) = 0$, that as mentioned in Step 2 will satisfy $|r_k| \le |a_n^{(n-1)} a_0|$. Then $P(x)$ will factor as $P(x) = (x - r_1)(x - r_2) \dots (x - r_n)$.

Now, since $P(x) = a_n^{(n-1)} p(\frac{x}{a_n})$ implies $p(x) = \frac{1}{a_n^{(n-1)}} P(a_n x)$, it follows,

$p(x) = \frac{1}{a_n^{(n-1)}} (a_n x - r_1) \dots (a_n x - r_n)$.

Or, $p(x) = a_n \left(x - \frac{r_1}{a_n}\right) \dots \left(a_n x - \frac{r_n}{a_n}\right)$.

Remark: Note that in the above algorithm I have assumed that common factoring already has been done. That is, I have assumed that there is no common integer factor (other than 1) between all the coefficients of a given original polynomial $a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$. In case there is one, and say there is a greatest common factor $C > 1$ between coefficients, then we will factor the polynomial as $C[a'_n x^n + a'_{n-1} x^{n-1} + \dots + a'_1 x + a'_0]$ and then apply the algorithm to the polynomial

$$p(x) = a'_n x^n + a'_{n-1} x^{n-1} + \dots + a'_1 x + a'_0$$

Proof: Given the original polynomial

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0,$$

as mentioned in the above remark, we can assume (without loss of generality) that the coefficients of $p(x)$ have no positive integer factor in common other than 1. Also, without loss of generality, we will assume that the leading coefficient a_n of $p(x)$ is a positive integer. Then the polynomial $P(x)$ defined in Step 1 of the algorithm is the same as

$$P(x) = x^n + c_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \dots + c_1 x + c_0$$
$$= a_n^{(n-1)} p(x/a_n)$$

where $c_k = a_n^{n-(k+1)} a_k$ for $k = 1, 2, \dots n$. On the other hand since zeros of $P(x)$ are just a_n times zeros of the polynomial $p(x)$, the latter polynomial will have a rational zero if and only if $P(x)$ has an integral zero. More precisely, an integer r will be a zero of $P(x)$ if and only if r/a_n a zero of the polynomial is $p(x)$.

Suppose now that, after using the integral zero theorem, the polynomial $P(x)$ can be factored as

$$P(x) = (x - r_1)(x - r_2) \dots (x - r_n)$$

where $r_1, r_2, \dots, r_n$ are all integers.

Now, as mentioned in Step 3 on this page, $P(x) = a_n^{(n-1)} p(\frac{x}{a_n})$ implies $p(x) = \frac{1}{a_n^{(n-1)}} P(a_n x)$, and therefore,

$p(x) = \frac{1}{a_n^{(n-1)}} (a_n x - r_1) \dots (a_n x - r_n)$,

and $p(x) = a_n \left(x - \frac{r_1}{a_n}\right) \dots \left(a_n x - \frac{r_n}{a_n}\right)$ is the required factorization of $p(x)$.