Equations of Angle Bisectors for Intersecting Lines

A.Astaneh, Vancouver, B.C.

Here is how you might formulize equations of the two angle bisectors of two intersecting lies on the coordinate plane.

Proposition: Given two intersecting lines with equations $L_1 : y = m_1 x + b_1$ and $L_2 : y = m_2 x + b_2$ on the $xy-plane$, $m_1 \neq m_2$, equations of the two angle bisectors of L_1 and L_2 are as follows:

$$(AB)_1 : y = \frac{m_2\sqrt{1+m_1^2} - m_1\sqrt{1+m_2^2}}{\sqrt{1+m_1^2} - \sqrt{1+m_2^2}} \left(x - \frac{b_1 - b_2}{m_2 - m_1} \right) + \frac{m_2 b_1 - m_1 b_2}{m_2 - m_1} .$$

$$(AB)_2 : y = \frac{m_2\sqrt{1+m_1^2} + m_1\sqrt{1+m_2^2}}{\sqrt{1+m_1^2} + \sqrt{1+m_2^2}} \left(x - \frac{b_1 - b_2}{m_2 - m_1} \right) + \frac{m_2 b_1 - m_1 b_2}{m_2 - m_1}$$

Proof: Since the point P of intersection of the two lines L_1 and L_2 has coordinated

$$P\left(\frac{b_1 - b_2}{m_2 - m_1} , \frac{m_2 b_1 - m_1 b_2}{m_2 - m_1} \right),$$

by a back and forth translation argument we can assume that the two lines intersect at the
origin; meaning we can assume without loss of generality that the two original lines L_1 and L_2 have equations $L_1 : y = m_1 x$ and $L_2 : y = m_2 x$, $m_1 \neq m_2$, and prove that equations of the two angle bisectors of L_1 and L_2 are

$$(AB)_1 : y = \frac{m_2\sqrt{1+m_1^2} - m_1\sqrt{1+m_2^2}}{\sqrt{1+m_1^2} - \sqrt{1+m_2^2}} \; x \quad ,$$

$$(AB)_2 : y = \frac{m_2\sqrt{1+m_1^2} - m_1\sqrt{1+m_2^2}}{\sqrt{1+m_1^2} - \sqrt{1+m_2^2}} \; x .$$

To this end we first note, in the particular case that $m_1 = -m_2$ the two lines L_1 and L_2 will be symmetric through the $y-axis$ in which one of the angle bisectors, $(AB)_1$, will be the $x-axis$, and the other one,

$(AB)_2$, the $y-axis$. Therefore in what follows we will assume that $m_1 \neq \pm m_2$.

To establish above equations for angle bisectors $(AB)_1$ and $(AB)_2$, it is enough to construct an isosceles triangle OMN with vertex at the origin $O(0,0)$ and the two equal sides OM and ON lying along the given lines L_1 and L_2 respectively. Because, having found coordinates of the point M and N on L_1 and L_2 respectively, slope of the line segment MN (that is m_{MN}) will be the same as the slope of one of the angle bisectors, and negative reciprocal of m_{MN} will be the slope of the other angle bisector.
More precisely, first choose an arbitrary point on L_1, say the point $M(1, m_1)$. Since
$|OM| = \sqrt{1+m_1^2}$, and since we have to choose a point $N(n, m_2 n)$ on the line $L_2 : y = m_2 x + b_2$ such that $|ON| = |OM|$, we must have $n^2 + (m_2 n)^2 = 1 + m_1^2$. Solving this latter equation for n, we get

$$n = \pm \frac{\sqrt{1+m_1^2}}{\sqrt{1+m_2^2}} .$$

Therefore we will have the following two choices for coordinates of N:

$$N_1\left(\frac{\sqrt{1+m_1^2}}{\sqrt{1+m_2^2}},\ \frac{m_2\sqrt{1+m_1^2}}{\sqrt{1+m_2^2}} \right) \quad \text{or} \quad N_2\left(\frac{-\sqrt{1+m_1^2}}{\sqrt{1+m_2^2}},\ \frac{-m_2\sqrt{1+m_1^2}}{\sqrt{1+m_2^2}} \right).$$

It is now easily seen that, the line segments MN_1 and MN_2 have the following slopes:

Slope of $MN_1 : \dfrac{m_2\sqrt{1+m_1^2} - m_1\sqrt{1+m_2^2}}{\sqrt{1+m_1^2} - \sqrt{1+m_2^2}}$, Slope of

$MN_2 : \dfrac{m_2\sqrt{1+m_1^2} + m_1\sqrt{1+m_2^2}}{\sqrt{1+m_1^2} + \sqrt{1+m_2^2}}$,

and proof of the Lemma is complete.

Corollary: If the slopes of two given lines $L_1 : y = m_1 x + b_1$ and $L_2 : y = m_2 x + b_2$ are reciprocal of each other, then their two bisectors are parallel to the bisectors of the system quadrants $y = \pm x$.

Proof: Just substitute $m_2 = 1/m_1$ in the above proposition to conclude the assertion.

Remark Since we know from elementary geometry that the two angle bisectors of intersecting lines are perpendicular, it follows that the slope on the right for MN_2 must be negative reciprocal of the slope on the left for MN_1. That is, we conclude the following identity which holds for any two nonzero numbers $m_1 \neq \pm m_2$ without any further algebra:

$$\frac{m_2\sqrt{1+m_1^2} - m_1\sqrt{1+m_2^2}}{\sqrt{1+m_1^2} - \sqrt{1+m_2^2}} = \frac{\sqrt{1+m_1^2} - \sqrt{1+m_2^2}}{m_1\sqrt{1+m_2^2} - \sqrt{1+m_1^2}}.$$

Example1 For the two lines with equations $L_1 : y = x$ and $L_2 : y = 2x$, $m_1 = 1$, $m_2 = 2$, and equations of the two angle bisectors are

$$(AB)_1 : y = \frac{2\sqrt{2} - \sqrt{5}}{\sqrt{2} - \sqrt{5}}\ x, \qquad (AB)_2 : y = \frac{2\sqrt{2} + \sqrt{5}}{\sqrt{2} + \sqrt{5}}\ x.$$

Example2 Consider the two lines with equations $L_1' : y = x + 3$ and $L_2' : y = 2x + 4$. Since these two lines are parallel to the respective lines L_1 and L_2 in Example1, their two angle bisectors $(AB)_1'$ and $(AB)_2'$ will have the same slopes as $(AB)_1$ and $(AB)_2$ in Example1 respectively. Also, since the two lines L_1' and L_2' intersect at the point $P(-1, 2)$ to obtain equations of $(AB)_1'$ and $(AB)_2'$ it is enough to write equations of the translations of the lines $(AB)_1$ and $(AB)_2$ to the point $P(-1, 2)$. Therefore we will have equations of the required angle bisectors $(AB)_1'$ and $(AB)_2'$ and as follows:

$$(AB)_1' : y = \frac{2\sqrt{2} - \sqrt{5}}{\sqrt{2} - \sqrt{5}}\ (x+1) + 2, \qquad (AB)_2' : y = \frac{2\sqrt{2} + \sqrt{5}}{\sqrt{2} + \sqrt{5}}\ (x+1) + 2.$$