The Proof Behind Kaprekar's "Game of 6174"

Ali Astaneh Ph.D. (Lon), Vancouver School Board, Vancouver, BC

In this article I will present my *algebraic* proof as for the whys and hows for the "mystery" behind Kaperkar's "Game of **6174**".

First, as a short history about the number **6174**, mostly known as Kaperkar's *constant* in math literature, I should mention that the Indian (recreational) mathematician D.R. Karperkar was the first to have discovered the magical properties of the number **6174**, and introduced it at a local conference in Madras, in 1949.

Peculiar as it may sound, the main reason I decided to pursue this subject further, as far as I did, was my personal affiliation with the year 1949!

I might also point out the reason I have preferred to use the term "Game of **6174**", rather than referring to **6174** as Kaperkar's *constant* is because I believe the whole process of starting with a random four-digit number and preforming a certain *operation* on it (and also to subsequent numbers obtained) repeatedly and eventually hit Kaperkar's *constant* **6174** after no more than seven attempts of the *operations* provides an amusing and mind simulating math game or activity at as early level as of late elementary/early secondary. That is, I think it provides an excellent practice for a deeper study of the roles digits play in a four –digit number, in most commonly used base 10. Indeed, the *algebraic* proof seen below may shed some lights on the roles of those digits, for secondary teachers as well!

Let it also be known that a proof by using computer programming, and examining all allowable 9990 four-digit selections in the game and eventually hitting **6174** (at most after seven *operations*) has already been established before, but I don't believe an *algebraic* proofs exists.

I would like to dedicate this proof to my son Sammie, as incidentally I concluded the main part of the proof on the evening of his birthday on June 14th 2020.

Having said all that; let's bring the rules and the promise of the game, first.

"Kapermark's Game of 6174"

Step 1: Choose any four-digit number ***N*** **=** **a b c d** you wish, with at least one digit different form the others, but allow leading zero four-digit numbers such as ***N*** **=** **0 b c d** in your selection.

Step 2: Arrange your selected four-digit number in ascending order first and then in descending order to obtain (most likely) two other four-digit numbers.

Step 3: Subtract the smaller of the four-digit numbers obtained in the Step 2 from the greater one obtain a new four-digit number, and call it $\boldsymbol{N_1}$.

Step 4: Repeat the combined *operations* of Steps 2 and 3 on the four-digit number $\boldsymbol{N_1}$ to obtained a next new number $\boldsymbol{N_2}$, and then $\boldsymbol{N_3}$, … .

The promise of the game is that performing the *operation* consisting of Steps 2 and 3 no more than 7 times you eventually hit the magic number **6174**. That is, one of the four-digit numbers $\boldsymbol{N_1}$, $\boldsymbol{N_2}$, … , $\boldsymbol{N_7}$ will eventually be equal to **6174** .

Interestingly, if your original selected number happens to be **6174** itself, the very first *operation* will produce the four-digit number **6174** itself, that is $\boldsymbol{N_1 = N}$. Or, mathematically speaking **6174** is the unique fixed number (point) of the *operation* in this game; and that is what makes Kaperkatr's "constant" so special.

Note that, according to what we do in Step 2 of the game, if two four-digit numbers happen to have the same digits, but in different order, then performing the *operation* on either of the two four-digit numbers will produce the same new four-digit number. I have used this fact quite a few number of times in the following proof. I should also point out that for the specific four-digit number $\boldsymbol{N = 8532}$ (or any other 4! four-digit number which has the same digits) we have $\boldsymbol{N_1 = 6174}$, that is only a single *operation* is needed to get to **6174**. I will use this fact in the following proof quite a few times as well. On the other extreme, there are also at least 48 four-digit numbers with, $\boldsymbol{N_7 = 6174}$; think of 4! = 24 of them being the four-digit numbers obtained by permuting the digits of the number $\boldsymbol{N}$ = 5213 in the following Example, plus another 4! = 24 for the four-digit numbers by permutations the digits of the number 5200, that the reader can verify.

Example: Choose the four-digit number $\boldsymbol{N}$ = 5213, and apply the *operation*. Then,
$\boldsymbol{N_1}$= 5321 – 1235 = 4086
$\boldsymbol{N_2}$= 8640 – 0468 = 8172
$\boldsymbol{N_3}$= 8721 – 1278 = 7443
$\boldsymbol{N_4}$= 7443 – 3447 = 3996
$\boldsymbol{N_5}$= 9963 – 3699 = 6264
$\boldsymbol{N_6}$= 6642 – 2466 = 4716
$\boldsymbol{N_7}$= 7641 – 1467 = **6174**

Theorem: Kaprekar's "Game of 6174":

Step 1: Choose any four-digit number $\boldsymbol{N}$ = **a b c d** you wish, with at least one digit different form the others, but allow leading zero four-digit numbers such as $\boldsymbol{N}$ = **0 b c d** , $\boldsymbol{N}$ = **0 0 c d** , or even $\boldsymbol{N}$ = **0 0 0 d** in your selection.
Step 2: Arrange your selected four-digit number in ascending order first andthen in descending order to obtain (most likely) two other four-digit numbers.
Step 3: Subtract the smaller of the four-digit number you obtained in Step 2 from greater one, and call the new four-digit number obtained $\boldsymbol{N_1}$= **A B C D**.
Step 4: Repeat the combined *operation* of **Steps 2&3** for the four-digit number $\boldsymbol{N_1}$= **A B C D** to obtained a next new four-digit number $\boldsymbol{N_2}$= **E F G H** and then $\boldsymbol{N_3}$, … .
Then, performing the *operation* consisting of **Steps 2&3** on your selected number no more that seven times, the resulting number of your last operation will be the four-digit number **6174**!

Proof (*Astaneh*): I will present the proof for the only two possible cases that I will label as **Case I** and **Case II**. The proof of **Case I** is short, whereas the proof for **Case II** will be much longer, and a bit tedious, as we need to bring in a second parameter in the argument (compared to the proof of the "Game of 495" in the previous Article 9), followed by a new kind of treatment.

So, let us first distinguish the two possible cases of a selection of a four-digit number; and prove them one by one.

Case I: After arranging the digits of your randomly selected four-digit number in ascending order, the resulting four-digit turns out to $\boldsymbol{N}$ = **a b b d**, with **a ≤ b< d** or **a ≤ b< d**.

Case II: After arranging the digits of your randomly selected four-digit number in ascending order, the resulting four-digit number turns out to be $\boldsymbol{N}$ = **a b c d** , with **a ≤ b < c ≤ b< d** .

Proof of case I: Assume after you arrange the digits of your randomly selected number it turns out to be $\boldsymbol{N}$ = **a b b d**, with **a ≤ b< d** or **a ≤ b< d**. Because of complete similarity of the proofs, I will assume **a ≤ b< d**. Let $\boldsymbol{e = d - a}$, then $\boldsymbol{e}$ can be any number **e = 1, 2, ... , 9**. And therefore, by Step 3 of the rules of the game after the first *operation*, the new four-digit number $\boldsymbol{N_1}$ will eventually be,
$\boldsymbol{N_1}$= **d b b a – a b b d = (1000 d +100 b +10b + a) – (1000 a +100 b +10b + d) = 1000 (d – a) + (a – d) 1000 e – e**. Hence, $\boldsymbol{N_1}$**= 999 e**, where **e = 1, 2, ... , 9**.
Therefore $\boldsymbol{N_1}$ can be only one of the following nine numbers,

999, **1998**, **2997**, **3996**, **4995**, **5994**, **6993**, **7992**, **8991**.

For convenience, I will examine all these number, but in the following order,

- **(i)** N_1= **1998** or **8991**. Since both these four-digit numbers have the same digits, by Step 2 of the rules for either case, N_2= **9981 – 1899 = 8082**, N_3=**8820 – 288 = 8532**, and N_4= 6174.
- **(ii)** N_1 = **2997** or **7992**. Again these numbers have the same digits, so in either case N_2= **9972 – 2799 = 7173**, N_3= **7731 – 1377 = 6354**, N_4= **6543 – 3456 = 3087**, N_5= **8730 – 378 = 8352**, and N_6= 6174.
- **(iii)** N_1 = **3996** or **6993**. Again these numbers have the same digits, so N_2= **9963 – 3699 = 6264**, N_3= **6642 – 2466 = 4176**. Since this number has the same as the target number **6174**, and the target number is the fixed number of the *operation* of the game it follows N_4= **6174**.
- **(iv)** N_1 = **999**. Then N_2= **9990 – 999 = 8991**. Since this latter number is one of the options for N_1 in case **(i)**, it follows that N_5= 6174.

This completes the proof of **Case I**.

Proof of Case II: The selected four digit number after arranging in ascending order will be of the form N = **a b c d**, where **a ≤ b < c ≤ d**. Let $e = d - a$ and $f = c - b$, then by Step 3 of the rules, after the first operation the new four-digit number N_1 can be expressed as,

N_1= **d c b a – a b c d = (1000 d +100 c +10b + a) – (1000 a +100 b +10c + d) =**

1000 (d – a) + 100 (c – b) + 10 (b – c) + (a – d) =

1000 (d – a) + 100 (c – b – 1) + 10 (10 + b – c) + (a – d) =

1000 (d – a) + 100 (c – b– 1) + 10 (9 + b – c) + (10 + a – d) =

1000 e + 100(f – 1) + 10(9 – f) + (10 – e). Hence,

N_1= **e (f – 1) (9 – f) (10 – e)**, where **1 ≤ f ≤ e ≤ 9** . **(2)**

Now we distinguish nine possible cases as follows,

$e = f,\ f+1,\ f+2,\ f+3,\ f+4,\ f+5,\ f+6,\ f+7,\ f+8.$

The rest of the proof would be to let e assume all above nine cases one by one, but the proof will be shorter if we cover the all those cases backwards, so here we go,

(A) $\underline{e = f + 8}$. Then **(2)** implies N_1= $\underline{(f+8)}$ $\underline{(f-1)}$ $\underline{(9-f)}$ $\underline{(2-f)}$. Now since $\underline{f+8}$ is a digit between **1** and **9** (inclusive), $f \leq 1$ which means we can only have $f = 1$, in which case, N_1= **9081**.
Then N_2= **9810 – 189 = 9621**, N_3= **9621 – 1269 = 8352**, and N_4= **<u>6174</u>** .

(B) $\underline{e = f + 7}$. Then **(2)** implies N_1= $\underline{(f+7)}$ $\underline{(f-1)}$ $\underline{(9-f)}$ $\underline{(3-f)}$. Again since $\underline{f+7}$ is a digit between **1** and **9**, we can only have $f = 1, 2$.

- <u>(i)</u> If $f = 1$, then N_1= **8082**. Then N_2= **8820 – 288 = 8532**, and N_3= **<u>6174</u>** .
- <u>(ii)</u> If $f = 2$, then N_1= **9171**. Then N_2= **9711 – 1179 = 8532**. Since this number has the same digits as N_1 in the previous part (A) above, it follows that N_3= **<u>6174</u>** .

(C) $\underline{e = f + 6}$. Then **(2)** implies N_1= $\underline{(f+6)}$ $\underline{(f-1)}$ $\underline{(9-f)}$ $\underline{(4-f)}$.
Since $\underline{f+6}$ is a digit and $1 \leq f$ we can only have $f = 1, 2, 3$.

- <u>(i)</u> If $f = 1$, then N_1= **7083**, N_2= **8730 – 378 = 8352**. Since this four-digit number is the same as N_2 in part (B)/<u>(ii)</u> above, we conclude that N_3 = **<u>6174</u>** .
- <u>(ii)</u> If $f = 2$, then N_1= **8172**, N_2= **8721 – 1278 = 7443**, N_3= **7443 – 3447 = 3996**, N_4= **9963 – 3699= 6264**. Since this four-digit number is the same as N_2 in **<u>Case I</u>** /**(ii)**, we conclude that N_6 = **<u>6174</u>** .
- <u>(iii)</u> If $f = 3$, then N_1= **9261**. Since this number has the same digits as N_2 in case (A) above, it follows that N_3 = **<u>6174</u>**.

(D) $\underline{e = f + 5}$. Then **(2)** implies N_1= $\underline{(f+5)}$ $\underline{(f-1)}$ $\underline{(9-f)}$ $\underline{(5-f)}$.
Since $\underline{f+5}$ is a digit and $1 \leq f$ we can only have $f = 1, 2, 3, 4$.

- <u>(i)</u> If $f = 1$, then N_1= **6084**, N_2= **8640 – 468 = 8172**. Since this four-digit number is the same as N_1 in above case (C)/<u>(ii)</u>, we conclude N_7 = **<u>6174</u>**.
- <u>(ii)</u> If $f = 2$, then N_1= **7173**. Since this four-digit number is the same as N_2 in the **<u>Case I</u>** /**(ii)**, it follows that N_5 = **<u>6174</u>**.
- <u>(iii)</u> If $f = 3$, then N_1= **8262**, N_2= **8622 – 2268 = 6354**. Since this four-digit number is the same as N_3 in **<u>Case I</u>** /**(ii)**, it follows that N_5 = **<u>6174</u>**.

(v) If $f = 4$, then N_1= **9351**, N_2= **9531 – 1359 = 8172**. Since this four-digit number is the same as N_1 in the case (C)/(ii), it follows that N_7 = **6174**.

(E) $e = f + 4$. Then **(2)** implies N_1= $(f + 4)$ $(f - 1)$ $(9 - f)$ $(6 - f)$. Since $f + 4$ is a digit and $1 \leq f$ we can only have $f = 1, 2, 3, 4, 5$.

(i) If $f = 1$, then N_1= **5085**, N_2= **8550 – 558 = 7992**. Since this number is the same as N_1 in **Case I /(ii)**, it follows that N_7 = **6174** .

(ii) If $f = 2$, then N_1= **6174**.

(iii) If $f = 3$, then N_1= **7263**, N_2= **7632 – 2367 = 5265**, N_3= **6552 – 2556 = 3996**. Since this four-digit number is also N_3 in case (C)/(ii), it follows that that N_6 = **6174** .

(iv) If $f = 4$, then N_1= **8352**. Since this number is the same as N_2 in case (C)/(i), it follows that that N_2 = **6174** .

(v) If $f = 5$, then N_1= **9441**, N_2= **9441 – 1449 = 7992**. Since this number is the same as N_2 in case (E)/(i), it follows that N_7 = **6174** .

(F) $e = f + 3$. Then **(2)** implies N_1= $(f + 3)$ $(f - 1)$ $(9 - f)$ $(7 - f)$.

Since $f + 3$ is a digit and $1 \leq f$ we can only have $f = 1, 2, 3, 4, 5, 6$.

(i) If $f = 1$, then N_1= **4086**, N_2= **8640 – 468 = 8172**. Since this four-digit number is the same a N_1 in case (C)/(ii), it follows that that N_7 = **6174** .

(ii) $f = 2$, then N_1= **5175**, N_2= **7551 – 1557= 5994** ,

N_3= **9954 – 4599=5355**, N_4= **5553 – 3555 = 1998**. Since this number is the same as N_1 in **Case I /(i)**, it follows that N_7 = **6174**.

(iii) If $f = 3$, then N_1= **6264**. Since this number is the same as N_2 in **Case I /(iii)**, it follows that N_3 = **6174** .

(iv) If $f = 4$, then N_1= **7353**, N_2= **7533** – **3357** = **4176**. Since this four-digit number has the same digits as the fixed number **6174** of the *operation* of the game, it follows that N_3= **6174**.

(v) If $f = 5$, then N_1= **8442**, N_2= **8442** – **2448** = **5994**. Since this number has the same digit as N_2 in case (ii) above, it follows that N_6 = **6174** .

(vi) If $f = 6$, then N_1= **9531**. Since this has the same digits as N_1 in the case (D) /(v), it follows that N_7 = **6174**.

(G) $e = f + 2$. Then **(2)** implies N_1= $(f+2)$ $(f-1)$ $(9-f)$ $(8-f)$.

Since $f + 2$ is a digit and $1 \leq f$ we can only have $f = 1, 2, 3, 4, 5, 6, 7$.

(i) If $f = 1$, then N_1= **3087**. Since this four-digit number is the same a N_4 in **Case I /(ii)**, it follows that N_3 = **6174**.

(ii) If $f = 2$, then N_1= **4176**. Again, since this four-digit number has the same digits as **6174**, the fixed number of *operation* of the game, it follows that N_2 = **6174**.

(iii) If $f = 3$, then N_1= **5265**. Since this number is the same as N_2 in case (E) /(iii), it follows that N_5 = **6174** .

(iv) If $f = 4$, then N_1= **6354**, and since this number is the same as as N_2 in in the case (D) /(iii) , it follows that N_5 = **6174**.

(v) If $f = 5$, then N_1= **7443**. Since this number is the same as N_2 in the case (C) /(ii), it follows that N_4 = **6174**.

(vi) If $f = 6$, then N_1= **8532**, and since this number has the same digits as N_1 in the case (E) /(iv), it follows that N_2 = **6174**.

(vii) If $f = 7$, then N_1= **9621**. Since this number is the same as N_2 in the case (A) at the start, it follows that N_4 = **6174**.

(H) $e = f + 1$. Then **(2)** implies N_1= $(f+1)$ $(f-1)$ $(9-f)$ $(9-f)$.

Since $f + 1$ is a digit and $1 \leq f$ we can only have $f = 1, 2, \ldots, 8$.

(i) If $\boldsymbol{f} = \mathbf{1}$, then $\boldsymbol{N_1}$= **2088**. Since this four-digit number has the same digits as $\boldsymbol{N_2}$ in case **Case I /(i)**, it follows that $\boldsymbol{N_3}$ = **6174**.

(ii) If $\boldsymbol{f} = \mathbf{2}$, then $\boldsymbol{N_1}$= **3177**. Since this number has the same digits as $\boldsymbol{N_1}$ in the case (D) /(ii), it follows that $\boldsymbol{N_5}$ = **6174** .

(iii) If $\boldsymbol{f} = \mathbf{3}$, then $\boldsymbol{N_1}$= **4266**. Since this number has the same digits as $\boldsymbol{N_2}$ in **Case I /(iii)**, it follows that $\boldsymbol{N_3}$ = **6174**.

(iv) If $\boldsymbol{f} = \mathbf{4}$, then $\boldsymbol{N_1}$= **5355**. Since this number is the same as $\boldsymbol{N_3}$ in the case (F) /(ii), it follows that $\boldsymbol{N_5}$ = **6174** .

Case I /(iv), it follows that $\boldsymbol{N_4}$ = **6174**.

(v) If $\boldsymbol{f} = \mathbf{5}$, then $\boldsymbol{N_1}$= **6444**, $\boldsymbol{N_2}$= **6444 – 4446 = 1998**. Since this number is the same as $\boldsymbol{N_1}$ in **Case I /(i)**, it follows that $\boldsymbol{N_5}$ = **6174**.

(vi) If $\boldsymbol{f} = \mathbf{6}$, then $\boldsymbol{N_1}$= **7533**. Since this number has the same digits as $\boldsymbol{N_1}$ in the case (F) /(iv) , it follows that $\boldsymbol{N_2}$ = **6174**.

(vii) If $\boldsymbol{f} = \mathbf{7}$, then $\boldsymbol{N_1}$= **8622**. Since this number has the same digits as $\boldsymbol{N_1}$ in the case (D) /(iii) , it follows that $\boldsymbol{N_5}$ = **6174**.

(viii) If $\boldsymbol{f} = \mathbf{8}$, then $\boldsymbol{N_1}$= **9711**, $\boldsymbol{N_2}$= **9711 – 1179 = 8532**. Since this number is the same as $\boldsymbol{N_1}$ in the case (G)/(vi), it follows that $\boldsymbol{N_3}$= **6174**.

(K) $\boldsymbol{e} = \boldsymbol{f}$. Then **(2)** implies $\boldsymbol{N_1}$= $\boldsymbol{f}\,(\boldsymbol{f}-\mathbf{1})\,(\mathbf{9}-\boldsymbol{f})\,(\mathbf{10}-\boldsymbol{f})$.

Since $\boldsymbol{f}$ is a digit and $\mathbf{1} \leq \boldsymbol{f}$ we can have $\boldsymbol{f} = \mathbf{1}, \mathbf{2}, \ldots, \mathbf{9}$.

(i) If $\boldsymbol{f} = \mathbf{1}$, then $\boldsymbol{N_1}$= **1089**. Since this number has the same digits as $\boldsymbol{N_1}$ in case (A), it follows that that $\boldsymbol{N_4}$ = **6174** .

(ii) If $\boldsymbol{f} = \mathbf{2}$, then $\boldsymbol{N_1}$= **2178**. Since this number has the same digits as $\boldsymbol{N_1}$ in the case (C) /(ii), it follows that $\boldsymbol{N_6}$ = **6174** .

(iii) If $\boldsymbol{f} = \mathbf{3}$, then $\boldsymbol{N_1}$= **3267**. Since this number has the same digits as $\boldsymbol{N_1}$ in the case (E) /(iii), it follows that $\boldsymbol{N_6}$ = **6174** .

(iv) If $\boldsymbol{f = 4}$, then $\boldsymbol{N_1}$= **4356**. Since this number has the same digits as $\boldsymbol{N_1}$ in the case (G) /(iv), it follows that $\boldsymbol{N_5}$ = **6174** .

(v) If $\boldsymbol{f = 5}$, then $\boldsymbol{N_1}$= **5445**, $\boldsymbol{N_2}$= **5544 – 4455 = 1089**. Since this number is the same as $\boldsymbol{N_1}$ in case (i) above, it follows that $\boldsymbol{N_5}$ = **6174**.

(vi) If $\boldsymbol{f = 6}$, then $\boldsymbol{N_1}$= **6534**. Since this number has the same digits as $\boldsymbol{N_1}$ in case (iv) above, it follows that $\boldsymbol{N_5}$ = **6174**.

(vii) If $\boldsymbol{f = 7}$, then $\boldsymbol{N_1}$= **7623**. Since this number has the same digits as $\boldsymbol{N_1}$ in the case (E) /(iii) , it follows that $\boldsymbol{N_6}$ = **6174**.

(viii) If $\boldsymbol{f = 8}$, then $\boldsymbol{N_1}$= **8712**. Since this number has the same digits as $\boldsymbol{N_2}$ in the case (D) /(i) , it follows that $\boldsymbol{N_6}$ = **6174**.

(ix) If $\boldsymbol{f = 9}$, then $\boldsymbol{N_1}$= **9801**. Since this number has the same digits as $\boldsymbol{N_2}$ in the case (v) above, it follows that $\boldsymbol{N_4}$ = **6174**.

Since we have covered all possible cases $\boldsymbol{e = f, f+1, f+2, f+3, f+4,}$ $\boldsymbol{f+5,\ f+6,\ f+7,\ f+8}$, the proof of **Case II** is also complete.