

## On Quadratic Mean of Positive Integers

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In statistics the **quadratic mean** of  $n$  given numbers  $x_1, x_2, \dots, x_n$  is defined to be the number

$$\sqrt{(x_1^2 + x_2^2 + \dots + x_n^2) / n} .$$

From number theory point of view, an elementary question arises whether for a set of positive integers  $x_1, x_2, \dots, x_n$  the quadratic expression above can be another positive integer. The affirmative answer to this question is easy to settle, and I leave this to the interested reader. A more challenging question would be whether the quadratic mean of a set of consecutive positive integers can be a positive integer. The answer to this question is also ( and not very surprisingly) positive, as the relation  $25^2 + 26^2 + \dots + 49^2 + \dots + 72^2 + 73^2 = (49 \times 51)^2$  implies

$$\sqrt{[(24 + 1)^2 + (24 + 2)^2 + \dots + (24 + 48)^2 + (24 + 49)^2]} / 49 = 51 ,$$

Which means the square mean of 49 consecutive integers is the integer 51.

In this short article I will show further that at least for three specific positive integers  $n$ , the square mean of a set  $n$  consecutive positive integers is the same as  $n$  itself, a very rare occurrence! In the mean time I first bring two propositions, useful for interested readers with a programmable hat (like I used to wear for very short very long ago!) to find more numbers  $n > 99359$ , enjoying this interesting property.

**Proposition 1:** The  $n$  consecutive positive integers  $N + 1, N + 2, \dots, N + n$  will have a quadratic mean of  $n$  if and only if the number  $N = \frac{1}{6}\sqrt{33n^2 + 3} - \frac{1}{2}(n + 1)$  is a positive integer.

**Proof:** The  $n$  consecutive positive integers  $N + 1, N + 2, \dots, N + n$  will have a quadratic mean of  $n$  if and only if  $\sqrt{\frac{(N + 1)^2 + (N + 2)^2 + \dots + (N + n)^2}{n}} = n$  , or if and only if

$$(N + 1)^2 + (N + 2)^2 + \dots + (N + n)^2 = n^3 .$$

Now, expanding the binomials, and making use of the popular identities  $1 + 2 + \dots + n = \frac{n(n+1)}{2}$  and  $1^2 + 2^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$  , and then dividing both sides of the resulting equation by  $n$  implies  $6N^2 + 6(n + 1)N + 1 + 3n - 4n^2 = 0$  . Since this is a quadratic equation with the solutions  $N = -\frac{1}{2}(n + 1) \pm \frac{1}{6}\sqrt{33n^2 + 3}$ , and since the only positive solution for  $N$  is

$$N = \frac{1}{6}\sqrt{33n^2 + 3} - \frac{1}{2}(n + 1), \quad (*)$$

and the proof is complete.

A second criteria for  $n$  consequent positive integers to have a square mean of exactly  $n$  is laid out in the following second criteria,

**Proposition 2:** A positive integer  $n$  will satisfy the relation of the form

$$(N + 1)^2 + (N + 2)^2 + \dots + (N + n)^2 = n^3, \text{ for some other positive integer } N,$$

**if and only if** the following conditions hold:

- (1)  $n$  is of the form  $n = 6k \pm 1$  for some positive integer  $k$ .
- (2) For the integer  $k$  in part (1) the integer  $33k^2 \pm 11k + 1$  is a complete square.

**Note:** Before bringing the proof I need to remind two things here; first note that conditions (1) and (2) above are both independent of  $N$ , but a property of  $n$  itself (useful for programming).

**Proof** The sufficiency part of the Lemma is easier to settle, so let's do that first and because of the complete similarity of the proofs I will only deal with the case that  $33k^2 + 11k + 1$  is a complete square and here is how it goes,

Since  $33k^2 + 11k + 1$  is assumed to be a complete square, let  $M = \sqrt{33k^2 \pm 11k + 1}$ . Then,

$$M^2 = 33k^2 + 11k + 1 = (6k + 1)^2 - k(3k + 1)$$

Or, 
$$M^2 + k(3k + 1) = (6k + 1)^2,$$

Or even better, 
$$(6k + 1)M^2 + k(3k + 1)(6k + 1) = (6k + 1)^3. \quad (**)$$

Next, using the popular identity  $1^2 + 2^2 + \dots + K^2 = \frac{K(K + 1)(2K + 1)}{6}$ , the reader

should be able to verify that (\*\*) is only a simplified version of the following relation

$$(M - 3k)^2 + \dots + (M - 1)^2 + M^2 + (M + 1)^2 + \dots + (M + 3k)^2 = (6k + 1)^3 \quad (**).$$

This completes the proof.

As I have pointed out earlier, the sufficiency of the criterion of the Proposition 2 follows, upon substitution  $N = M - 3k - 1$ .

Proposition 2 can be used through softwares to search for  $n$ 's satisfying conditions. Early 2010's I used Maple software and I found three  $n$ 's satisfying conditions as seen below,

If we choose  $n = 47$ , (\*) implies  $N = 21$ , and we get

$$(21 + 1)^2 + (21 + 2)^2 + \dots + (21 + 47)^2 = 47^3,$$

or 
$$22^2 + 23^2 + \dots + 68^2 = 47^3. \quad (1) \text{ Hence}$$

$\sqrt{(22^2 + 23^2 + \dots + 68^2)/47} = 47$  which means the quadratic mean of 47 consecutive positive integers is 47 itself.

Note, this also means the square means of 47 consecutive negative integers is 47. For,

$$(-68)^2 + (-67)^2 + \dots (-23)^2 + (-22)^2 = 47^3.$$

**Remark:** If you give up the stiffer condition of consecutiveness for the positive integers and ask for the more relax condition of distinctiveness of positive integers you can also have more sets of 47 positive integers with quadratic mean 47. For one reason, this is because the sum of squares of two specific numbers in the original finite set  $\{22, 23, \dots, 68\}$  of the 47 consecutive numbers could be equal to sum of the squares of two other positive integers which may not be in the set at all. For example since  $45^2 + 61^2 = 11^2 + 75^2$ , you could replace the numbers 45 and 61 by 11 and 75 to have the set  $\{11, 22, 23, \dots, 44, 46, 47, \dots, 60, 62, 63, \dots, 68, 75\}$  of 47 distinct positive integers with quadratic mean 47. That is, you could have

$$\sqrt{(11^2 + 22^2 + \dots + 44^2 + 46^2 + \dots + 60^2 + 62^2 + \dots + 68^2 + 75^2)/47} = 47.$$

For another reason, you can use Pythagorean Theorem twice to get a third set of 47 numbers with the same quadratic mean. More precisely you could first substitute  $15^2 + 20^2$  for  $25^2$  in (1), thereby increasing cardinality of the original set  $\{22, 23, \dots, 68\}$  by one first, but then substitute  $85^2$  for  $51^2 + 64^2$  to reduce the cardinality back to original 47. So our third set of 47 distinct positive integers  $\{15, 20, 22, 23, \dots, 50, 52, 53, \dots, 67, 85\}$  will also have a quadratic mean 47, which means the square root,

$$\sqrt{\frac{15^2 + 20^2 + 22^2 + 23^2 + 24^2 + 26^2 + \dots + 50^2 + 52^2 + \dots + 60^2 + 62^2 + \dots + 68^2 + 85^2}{47}}$$

is also exactly equal to 47. Yet, a fourth such set of non-consecutive 47 positive integers with square mean 47 can be obtained if you again substitute  $15^2 + 20^2$  for  $25^2$  but at the same time substitute for  $80^2 - 64^2$  for  $85^2$  for  $48^2$  in (1).

Having settled with 47, the next number  $n$  for which the relation

$$\sqrt{\frac{(N+1)^2 + (N+2)^2 + \dots + (N+n)^2}{n}} = n, \quad (2)$$

Would be the integer  $n = 2161$  (and there will be no other  $47 < n < 2161$  satisfying (1) according to Maple software. If you substitute  $n = 2161$  in (\*) you get  $N = 21$ , which means  $(988 + 1)^2 + (988 + 2)^2 + \dots + (988 + 2161)^2 = 2161^3$

or 
$$989^2 + 990^2 + \dots + 3149^2 = 2161^3.$$

Hence the quadratic mean of the set  $\{989, 990, \dots, 3149\}$  of the 2161 consecutive numbers is 2161.

Again the same Remark I made above for  $n = 47$  applies to this second integer  $n = 2161$ . That is, there are other sets of 2161 distinct but not consecutive positive numbers derived from the set of consecutive numbers  $\{989, 990, \dots, 3149\}$  with the same quadratic mean 2161. More

precisely, since  $2250^2 + 3050^2 = 550^2 + 3750^2$ , if you replace the integers 2250 and 3050 by 550 and 3750 in the set  $\{989, 990, \dots, 3149\}$  you obtain the following set of distinct 2161 integers with quadratic mean 2161,

$$\{550, 989, 990, \dots, 2249, 2251, 2252, \dots, 3049, 3051, 3052, \dots, 3149, 3750\}.$$

I let the interested reader to find a third set of distinct but not consecutive sets of 2161 numbers with quadratic mean 2161 by double use of Pythagorean Theorem, just as I did for the case of  $n = 47$  in above Remark.

The third promised number  $n$  satisfying (2) for a specific positive integer  $n$  is 99359 (and again there will be no other  $n$  between 2161 and 99359 will satisfy (2) for any positive integer  $n$ ). This is because if you substitute 99359 in (\*) for  $n$  you get  $= 45449$ , and therefore we have

$$(45449 + 1)^2 + (45449 + 2)^2 + \dots + (45449 + 99359)^2 = 99359^3,$$

or 
$$45450^2 + 45451^2 + \dots + 144808^2 = 2161^3.$$

Another words we have  $(45449 + 1)^2 + (45449 + 2)^2 + \dots + (45449 + 99359)^2 = 99359^3$ .

Again, as mentioned in the Remark for the case of  $n = 47$  you can have more sets of 99359 distinct but not consecutive positive integers with quadratic mean 99359. This time let us try to find one by double application of Pythagorean Theorem. Since  $30000^2 + 40000^2 = 50000^2$ , we first remove the integer 50000 from the set  $\{45450, 45451, \dots, 144808\}$  and replace it with the two integers 30000 and 40000 which will increase the cardinality of the set by 1. Then, since  $138030^2 + 144000^2 = 199470^2$  we remove the two integers 138030 and 144000 from the new set and substitute them with 199470 and get the following set of 99359 distinct numbers with quadratic mean 99359,

$$\{300000, 40000, 45450, \dots, 49999, 50001, \dots, 138029, 138031, \dots, 143999, 144001, \dots, 144808, 199470\}.$$

I leave it to the reader to find a third set of distinct 99359 integers with quadratic mean 99359 by replacing two members of the set with other two integers which aren't in the set.

As I have mentioned according to Maple software  $n = 47, 2161, 99359$  are the only first three existing positive integers  $n$  for which square mean of  $n$  consecutive positive integers happen to be  $n$  itself. What, in particular, makes 47 more peculiar among the above three integers is that, not only the sum of squares of 47 consecutive positive integers can be a cubic, but also the sum of squares of 47 consecutive positive integers can be a square, simply because

$$(730 + 1)^2 + (730 + 2)^2 + \dots + (730 + 47)^2 = 5170^2 = 110^2 \cdot 47^2,$$

or

$$731^2 + 732^2 + \dots + 777^2 = 5170^2,$$

In statistical terms, this last relation means the quadratic mean of 47 consecutive numbers can also be a different integer than 47 itself; this time 110. This again strengthens my earlier conjecture that 47 is the *most peculiar* positive integer.