n Times Differentiable Interpolating Curves and the Pascal Triangle

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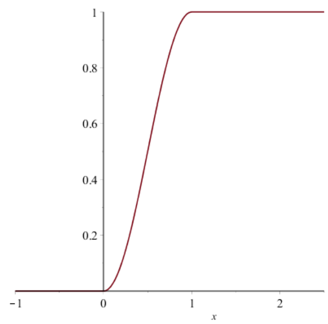
Let us first consider the point of view that the function defined by

$$f(x) = \begin{cases} 0 & x \leq 0 \\ \frac{1}{2}\cos\ \pi(x+1) + \frac{1}{2} & 0 < x < 1 \\ 1 & x \geq 1 \end{cases}$$



is an entirely differentiable function over the real number line, in the sense of having a graph with no corner/sharp points. However, as it can easily it can be observer its derivative

$$f'(x) = \begin{cases} 0 & x \leq 0 \\ -\frac{\pi}{2}\sin\ \pi(x+1) & 0 < x < 1 \\ 0 & x \geq 1 \end{cases}$$

isn't differentiable neither at $x = 0$, nor at $x = 1$. This means $f''(x)$ isn't define at numbers $= 0,1$, which means the graph of $f''(x)$ defined as,

$$f''(x) = \begin{cases} 0 & x < 0 \\ -\frac{\pi^2}{2}\cos\ \pi(x+1) & 0 < x < 1 \\ 0 & x > 1 \end{cases}$$

isn't defined at $x = 0,1$ and with two jumps at those two numbers.

The question then arises whether there are interpolation curves between half lines defined by $y = 0,\ x \leq 0$, and$y = 1,\ x \geq 1$, that are two, three, four, ... , or even *n* times (entirely) differentiable for an arbitrary $n = 2,3,4, \ldots$. As my following proposition will show shortly the answer to this question is affirmative, and a <u>polynomial</u> function of order $2n+1$ will be the desired *n* times differentiable interpolation. And what is more, and perhaps even striking, is that the popular Pascal Triangle can be used to derive an algorithm to define those *n* times differentiable functions of degree $2n+1$ interpolating between the two horizontal half lines $y = 0,\ x \leq 0$, and $y = 1,\ x \geq 1$. And here is the Proposition.

<u>Proposition</u>: For any positive integer n, the piece-wise function

$$f_n(x) = \begin{cases} 0 & x \le 0 \\ x^{n+1} \cdot \sum_{j=0}^{n} (-1)^{j} {}_{n+j}C_j \, (x-1)^j & 0 < x < 1, \\ 1 & 1 \le x \end{cases}$$

is an entirely n times differentiable function interpolating the horizontal half lines $y = 0,\ x \le 0$, and $y = 1,\ x \ge 1$, where ${}_{n+j}C_j$ means combinations of $n+j$ object taken j at a time .

Proof: It is obvious that for each n the function $f_n(x)$ is an entirely continuous at the two points $(0,0)$ and $(1,1)$, and hence entirely. The proof that $f_n(x)$ is n times differentiable at the two points $(0,0)$ and $(1,1)$ is accomplished by induction on n (a kind of similar to the proof of the article "General Product Rule" presented in Calculus 2 section of this website. I leave the proof an exercise to the enthusiastic reader. I might point out in the process a familiarity about identities among combinations ${}_{n+j}C_j$ for different positive integers $n+j$ and j's is required , and change if indices in sigmas is required.

Connection with the Pascal Traiangle: As the reader can check from the underlined entries of the Pascal Triangle below, the coefficients in each sigma $\sum_{j=0}^{n}(-1)^j \, {}_{n+j}C_j \, (x-1)^j$ in the definition of the n times differentiable function $f_n(x)$ in the above Proposition are just the first $n+1$ entries in (downward to the right) of the $(n+1)^{th}$ diagonal of the triangle, but alternated in the sign (Note that the usual convention ${}_{n+j}C_j = 0$ when $j = 0$ is applied here). For example, the coefficients in the sigma defining $f_1(x)$ are 1 and – 2 , the coefficients in the sigma defining $f_2(x)$ are 1 and -3 and 6, and the coefficients in the sigma defining $f_3(x)$ are 1 and -4 and 10 and -20, as I have underlined them in the Pascal Triangle below show.

1

<u>1</u> **1**

<u>1</u> **<u>2</u>** **1**

<u>1</u> **<u>3</u>** **3** **1**

<u>1</u> **<u>4</u>** **<u>6</u>** **4** **1**

1 **<u>5</u>** **<u>10</u>** **10** **5** **1**

1 **6** **<u>15</u>** **<u>20</u>** **15** **6** **1**

1 **7** **21** **<u>35</u>** **35** **21** **7** **1**

1 **8** **28** **56** **<u>70</u>** **56** **28** **8** **1**

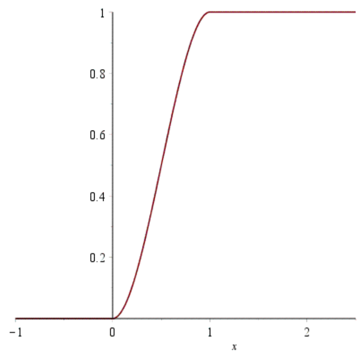
As examples how the graphs of the first four functions for $f_1(x)$, $f_2(x)$, $f_3(x)$ and $f_4(x)$ look like I have graph them below. Note that in each example the sigma in the proposition has been simplified after the necessary expansions.

Example 1 By the above Proposition $f_1(x)$ amounts to,

$$f_1(x) = \begin{cases} 0 & x \leq 0 \\ x^2[1-2(x-1)] & 0 < x < 1 \\ 1 & x \geq 1 \end{cases}$$

> $f1 := x \rightarrow piecewise\left(x < 0, 0, x < 1, x^2.(3-2x), x < 3, 1\right);$

> $plot(f1(x), x = -1..2.5);$



Note that, as expected, the derivative $f_1'(x)$ is entirely continuous because

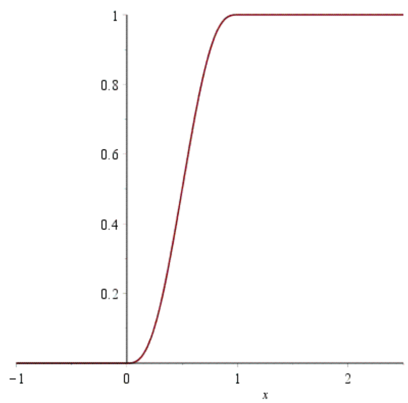
$$\frac{d}{dx}\left(x^2\,(1-2\,(x-1))\right) = -6\,(x-1)\,x$$

Example 2 The Proposition determines $f_2(x)$ as,

$$f_2(x) = \begin{cases} 0 & x \leq 0 \\ x^3[1-3(x-1)+6(x-1)^2\,] & 0 < x < 1 \\ 1 & x \geq 1 \end{cases}$$

> $f2 := x \rightarrow piecewise\left(x < 0, 0, x < 1, x^3.\left(6x^2 - 15x + 10\right), x < 3, 1\right);$

> $plot(f2(x), x = -1..2.5);$



Note that the derivative $f_2''(x)$ is entirely continuous because

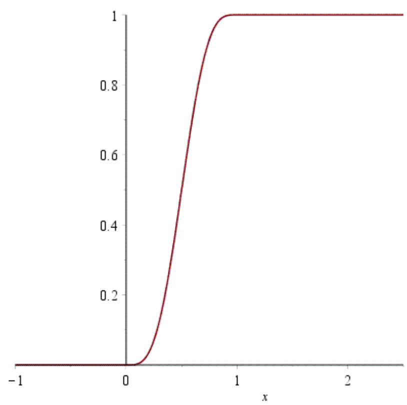
$$\frac{d^2}{dx^2}\left(x^3\left(1-3(x-1)+6(x-1)^2\right)\right)=60\,x\left(2\,x^2-3\,x+1\right)$$

Example 3 The Proposition implies $f_3(x)$ is,

$$f_3(x)=\begin{cases}0 & x\le 0\\ x^4[1-4(x-1)+10(x-1)^2-20(x-1)^3\,] & 0<x<1\\ 1 & x\ge 1\end{cases}$$

> $f3 := x \rightarrow piecewise\left(x<0, 0, x<1, x^4.\left(-20\,x^3+70\,x^2-84\,x+35\right), x<3, 1\right);$

> $plot(f3(x), x=-1..2.5);$



Note that the derivative $f_3^{(3)}(x)$ is entirely continuous because

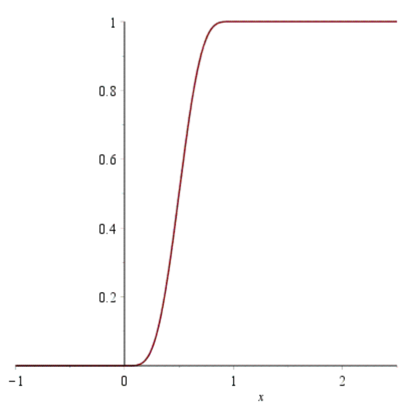
$$\frac{d^3}{dx^3}\left(x^4\left(1-4(x-1)+10(x-1)^2-20(x-1)^3\right)\right)=-840\,x\left(5\,x^3-10\,x^2+6\,x-1\right)$$

Example 4: When $n=4$, the Proposition implies $f_4(x)$ as follows,

$$f_4(x)=\begin{cases}0 & x\le 0\\ x^5(70x^4-315x^3+540x^2-420x+126) & 0<x<1\\ 1 & x\ge 1\end{cases}$$

> $f4 := x \rightarrow piecewise\left(x<0, 0, x<1, x^5.\left(70\,x^4-315\,x^3+540\,x^2-420\,x+126\right), x<3, 1\right);$

> $plot(f4(x), x=-1..2.5);$



Note that, as expected, the derivative $f_4^{(4)}(x)$ is entirely continuous because

$$\frac{d^4}{dx^4}\left(x^5\left(1-5(x-1)+15(x-1)^2-35(x-1)^3+70(x-1)^4\right)\right)=$$
$$15\,120\,x\left(14x^4-35x^3+30x^2-10x+1\right)$$

Observe, say from the last two Examples, that the higher order differentiability of $f_4(x)$ (compared to graph of $f_3(x)$) is reflected by the fact that that graph of $f_4(x)$ looks "flatter" to the right vicinity of $x=0$, and the left vicinity of $x=1$.

Remark: To verify that the derivatives $f_5^{(5)}(x)$, $f_3^{(6)}(x)$, and the derivative $f_7^{(7)}(x)$ will also be entirely continuous, it is enough to consider that

$$\frac{d^5}{dx^5}\left(x^6\left(1-6(x-1)+21(x-1)^2-56(x-1)^3+126(x-1)^4-252(x-1)^5\right)\right)=$$
$$-332\,640\,x\left(42x^5-126x^4+140x^3-70x^2+15x-1\right)$$

$$\frac{d^6}{dx^6}\left(x^7\left(1-7(x-1)+28(x-1)^2-\right.\right.$$
$$\left.\left.84(x-1)^3+210(x-1)^4-462(x-1)^5+924(x-1)^6\right)\right)=$$
$$8\,648\,640\,x\left(132x^6-462x^5+630x^4-420x^3+140x^2-21x+1\right)$$

$$\frac{d^7}{dx^7}\left(x^8\left(1-8(x-1)+36(x-1)^2-120(x-1)^3+\right.\right.$$
$$\left.\left.330(x-1)^4-792(x-1)^5+1716(x-1)^6-3432(x-1)^7\right)\right)=$$
$$-259\,459\,200\,x\left(429x^7-1716x^6+2772x^5-2310x^4+1050x^3-252x^2+28x-1\right)$$

And finally, I leave it to the interested reader to use function transformations to conclude the following Theorem, as a generalization of the above Proposition.

Theorem: For any positive integer n, the function

$$f_n(x)=\begin{cases} b & x\le a \\ \dfrac{d(x-a)^{n+1}}{(c-a)^{n+1}}\left[\,1+\displaystyle\sum_{j=1}^{n}(-1)^j\,{}_{n+j}C_j\,(x-c)^j\,\right] & a<x<c \\ d & x\ge c \end{cases}$$

defines an entirely n times differentiable interpolation from the right end of the half horizontal line interpolating the half lines $y=0,\ x\le 0$, and $y=1,\ x\ge 1$.