Tangent Lines and Secant Segments for Parabolas and Cubic Curves

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The main objective of this article is first to present a canonical way to parametrize all possible secant line segments supported by any parabola or any cubic curve, in each case as a two parameter family of real numbers. And then use a particular member from each family to find the slope of the tangent line to any pre-assigned point on any parabola or cubic curve and express equations of the tangent lines at the BC Pre-Calculus 11 & 12 curriculum level in Canada. Each secant segment in either case is described in terms of coordinates of its two end points, expressed in terms of the two selected parameters.
As mentioned earlier this will enable BC Pre-Calculus 11 & 12 students to write down equations of a tangent lines to parabolas or cubic curves at any specific point, avoiding Calculus, but only by considering coordinates of the end points of a specific secant segment in each two parameter family.

Remind in passing that a common way to parametize secant segments for any curve defined by $y = f(x)$ would be to describe the family of all secants by considering the possible two end points secants segments as $\{(s, f(s))\ , B\big(t, f(t)\big) : s < t \in R\}$, but that isn't the avenue taken in this note.

Let us first start with a general curve defined by a function $y = f(x)$. Then one way to go about describing all secant segments would be first to choose $u \in R$ as a first parameter, and consider the point $M(u, f(u))$ on the graph with the tangent line T_M to the graph at this point. Then, as the second parameter select $p > 0$ and consider the point $A(u-p, f(u-p))$ on the curve $y = f(x)$. And then try to locate a second point $B\big(x, f(x)\big)$ on the curve such that the secant segment AB is parallel to the tangent line T_M. To this end, we equate the slope of the line segment AB and the tangent line T_M as,

$$\frac{f(x) - f(u-p)}{x - (u-p)} = f'(u).$$

We now return to our specific cases of our interest, that is parabolas and cubic curves.. First let us consider the case of a parabola, say represented by $f(x) = ax^2 + bx + c$. Then the following should be an straightforward exercise for any Calculus student.

Exercise : Show that for any function $f(x) = ax^2 + bx + c$ the above equation has the solution $x = u + p$, in terms of our selected parameters u and p.

The assertion in this exercise implies that all secant segments AB on the parabola parallel to T_M have end points $A\big(u-p, f(u-p)\big)$ and $B\big(u+p, f(u+p)\big)$ for some $p > 0$. On the other hand the Mean Value Theorem of Calculus guarantees that any secant line segment on the parabola is parallel to some T_M . Therefore we have proved the following,

Proposition 1: The two parameter family of line segments

$$\{A\big(u-p, f(u-p)\big), B\big(u+p, f(u+p)\big)\}\ , u \in R, p > 0,$$

describes all possible secant line segments for any Parabola $y = f(x)$.

An immediate advantage of the description in above proposition is the following corollary, which enables students at of at grade 11 math level to find equations of tangent lines to a parabola avoiding derivatives from Calculus.

Corollary 1: Given a parabola represented by $f(x) = ax^2 + bx + c$ and a point $M(u, f(u))$ on it, in order to find the equation of the tangent line T_M to the parabola at M first consider the slope of secant line with the two end points $A\big(u-1, f(u-1)\big)$ and $B\big(u+1, f(u+1)\big)$ to be the slope of T_M, and then write equation of T_M as,

$$y - f(u) = \frac{f(u+1) - f(u-1)}{2}\ (x-u).$$

Example 1 Find equation for the tangent line T_M to parabola $f(x) = x^2 + 2x$ at the point $M(2{,}8)$ on its graph.

Solution Since $u = 2$,$f(u) = 8$, $f(u-1) = f(1) = 3, f(u+1) = f(3) = 15$, and $f(u-1) = f(1) = 3$, the slope of the tangent line T_M is $\frac{f(u+1)-f(u-1)}{2} = \frac{15-3}{2} = 6$, and the required equation is $y - 8 = 6\ (x-2)$.

Also, note that Proposition 1 describes the two end points of all possible secant line segments for the parabola $f(x) = x^2 + 2x$ in the example as,
$\{A(u-p, (u-p)^2 + 2(u-p), B(u+p, (u-p)^2 + 2(u-p)\ ) \colon u \in R, p > 0\}$.

In the case of a cubic curve, as one might expect, the algebra involved in describing possible secant line segments would be a bit more challenging. Given a cubic function $g(x) = ax^3 + bx^2 + cx + d$, first I might recall the popular fact that any such function, upon a vertical scaling by $\frac{1}{a}$ followed by an horizontal shift by the abscissa $-b/3a$ of its inflection point can be converted into the more convenient form $f(x) = x^3 + \alpha x + \beta$. In other words $f(x) = \frac{1}{a}\ g\left(x - \frac{b}{3a}\right) = x^3 + \alpha x + \beta$, in which case the coordinates of the inflection point for $f(x)$ will be just $P(0, \beta)$. This inflection will play a significant role in our description of the secant segments because it also is a point of symmetry for the cubic curve $y = f(x)$. So, I will first describe possible secant segments for such $f(x)$, because then possible secant segments for $g(x)$ can be found by a back-horizontal shifting by $b/3a$ followed by a vertical scaling by a of those for of $f(x)$.

To find possible secant segments for $f(x) = x^3 + \alpha x + \beta$ I first recall the fact that for the point of inflection $P(0, \beta)$ of the cubic curve there will be no secant line segment parallel to the tangent line T_P at P (see item # 13 on Calculus section of the website for proof). However for any other point $M(u, f(u))$ with $u \neq 0$ we can, and will, describe coordinates of the end points for all possible secant segments parallel to T_M. To this end, observe that because the inflection point $P(0, \beta)$ is at the same time the point of symmetry for the curve, it would suffice to consider our first parameter to be $0 < u$ and as the algebra will show shortly, it is also enough to choose our second parameter to be any real number $-u \leq p \leq 3u$. As before let T_M be the tangent line at $M(u, f(u))$ to the cubic curve and consider $A\big(u-p, f(u-p)\big)$ as one of the two end points of a secant segment parallel to T_M . Then to locate the other possible end point $B\big(x, f(x)\big)$ of a secant segment AB parallel to T_M we need to equate slopes of AB and T_M as follows,

$$\frac{f(x)-f(u-p)}{x-(u-p)}=\frac{x^3+\alpha x+\beta-(u-p)^3-\alpha(u-p)-\beta}{x-(u-p)}=3u^2+\alpha.$$

This equation simplifies into $x^2+(u-p)x+(u-p)^2-3u^2=0$ with solutions:

$$x_1=\frac{(p-u)}{2}+\frac{\sqrt{3}}{2}\sqrt{4u^2-(p-u)^2}\ ,x_2=\frac{(p-u)}{2}-\frac{\sqrt{3}}{2}\sqrt{4u^2-(p-u)^2}\ .$$

Note that our earlier assumption $-u \le p \le 3u$ guarantees that the discriminant of the above quadratic equation is non-negative, so that x_1 and x_2 are distinct real numbers, unless $p=-u$ or $p=3u$, in which case we will have a double root $x_1=x_2$.

Indeed if $p=3u$, then $x_1=x_2=u$, which means the two end points of the secant segment parallel to T_M are $A\big(-2u, f(-2u)\big)$ and $M(u, f(u))$ itself. This makes sense because the tangent line T_M is bound to intersect the cubic curve at another point anyway, so that *AM* is at the same time a secant segment and a segment from tangent line T_M .
On the other hand for $p=-u$ we get $x_1=x_2=-u$, that is this time $A\big(2u, f(2u)\big)$ and $B(-u, f(-u))$ will be the end points of the secant segment parallel to T_M . Observe that the coordinates for the end points of the secant segment *AB* show that *AB* is just the reflection of the previous secant *AM* through the point of symmetry $P(0,\beta)$ of the curve.

Next, since for $-u<p<3u$ we have two distinct real numbers x_1 and x_2 as the solutions of the quadratic equation, we will have two possible second end points as $B_1(x_1, f(x_1))$ and $B_2(x_2, f(x_2))$, and therefore we obtain <u>three</u> possible secant segments AB_1, AB_2, and B_1B_2 parallel to T_M for any $-u<p<3u$. This completes description of all possible secant segments of the cubic curve that are parallel to T_M .
Again, since the Mean Value Theorem of Calculus guarantees that any secant line segment to the curve is parallel to some T_M , and since $P(0,\beta)$ is a point of symmetry for the curve so that the case $u<0$ is unnecessary to be considered, the following proposition canonically describes all possible secant segments for the cubic curve.

However, before the formal assertion of the proposition, because of later application in Example 2 below it is worth a while also to consider the notable case of $p=u$, as we will end up with $x_1=\sqrt{3}u$ and $x_2=-\sqrt{3}u$ in which case coordinates of the first end point become $\big(u-p, f(u-p)\big)=A(0,\beta))=P(0,\beta)$, which means A is just the point of symmetry of the cubic curve. Considering also that coordinates of the two other possible end points for secants will be $B_1(\sqrt{3}u, f\big(\sqrt{3}u\big))$ and $B_2(-\sqrt{3}u, f\big(-\sqrt{3}u\big))$, it follows the extended secant segment $B_1\,B_2$ will be <u>a line of symmetry</u> of the curve.

<u>Proposition 2</u>: Let $f(x)=x^3+\alpha x+\beta$ be a cubic curve and for all possible real numbers satisfying $u>0$, $-u \le p \le 3u$, let

$$x_1=\tfrac{(p-u)}{2}+\tfrac{\sqrt{3}}{2}\sqrt{4u^2-(p-u)^2}\ \text{ and }\ x_2=\tfrac{(p-u)}{2}-\tfrac{\sqrt{3}}{2}\sqrt{4u^2-(p-u)^2}$$

If we denote all possible points on the cubic curve with above abscissa by $B_1(x_1, f(x_1))$ and $B_2(x_2, f(x_2))$, then the two parameter family

$$\{AB_1,\ AB_2,\ B_1B_2 : u > 0\ ,\ -u \le p \le 3u\ \}$$

is a canonical description of all possible secant line segments for the cubic curve.

Note that, the reason the case $u < 0$ isn't necessary to be included in description of possible secant line segments in above Proposition 2 is that, the point $P(0, \beta)$ being a point of symmetry for the cubic curve, the tangent lines T_M and T_N with $N(-u, f(-u))$ are parallel, and thus every secant segment parallel to T_M will be parallel to T_N as well.

Next, parallel to Corollary 1 for parabolas, we assert the following corollary that enables students to find equation of the tangent line to a cubic curves at a pre-assigned point at a Pre-Calculus level, even avoiding long division of polynomials (see Examples 2 and 3).

Corollary 2: Given a cubic function represented by $f(x) = x^3 + \alpha x + \beta$ and a point $M(u, f(u)),\ u \neq 0,$ to find the slope of the tangent line T_M at Pre-Calculus level, consider the slope of the secant line segments with the two end points $A(0, \beta)$ and $B\left(\sqrt{3}u, f(\sqrt{3}u)\right)$ the slope of T_M . If $u = 0$ just consider α to be the slope of T_M .

Example 2(a) Find an equation for tangent line T_M to the cubic curve

$$f(x) = x^3 - 9x + 4$$

at the point $M(2, -6)$ on the graph.

Solution Since $u = 2,\ f(2) = -6,$ by above Corollary 2 we will find the slope of the secant segment with the end points $A(0, \beta) = A(0,4)$ and $B\left(2\sqrt{3}, f(2\sqrt{3})\right)$ as the slope of T_M . Since $f(2\sqrt{3}) = (2\sqrt{3})^3 - 9(2\sqrt{3}) + 4 = 6\sqrt{3} + 4$, the end point B has coordinates $B(2\sqrt{3}, 6\sqrt{3} + 4),$ and the slope of T_M is $\frac{6\sqrt{3}+4-4}{2\sqrt{3}-0}$ = 3, and the required equation for T_M will be $y + 6 = 3(x - 2)$, or $y = 3x - 12$.

Remark: In our last example, shortly seen below we will make use of the popular known fact that for any cubic function $g(x) = ax^3 + bx^2 + cx + d$, a horizontal shift by $\frac{-b}{3a}$ followed a vertical scaling of $\frac{1}{a}$ will eliminate the x^2 term in the expression defining $g(x)$ is eliminated and the expression defining the "transformed " function. Therefore we end up with a more convenient expression which happens to be of our desired form $f(x) = x^3 + \alpha x + \beta$ to which Proposition 2 and Corollary 2 can be applied.efficient to 1. More precisely, the transformed function $f(x) = \frac{1}{a} g\left(x - \frac{b}{3a}\right)$ will be of the ideal form $f(x) = x^3 + \alpha x + \beta$ as being necessary in Proposition 2 and Corollary 2.

Example 2(b) Find an equation for tangent line T_N to the cubic curve

$$f(x) = x^3 - 9x + 4$$

at the point $N(-2, -14)$ on the graph.

Solution Since the cubic curve has the inflection point $P(0,4)$, and since this point is also the point of symmetry for the curve, the tangent lines T_N and and T_M with $M(2,-6)$ are parallel and thus have the same slope. Since in Example 2(a) we found the slope of T_M to be 3, this will also be the slope for T_N . Therefore the equation for T_N will be,
Be $y+14=3(x+2)$, or $y=3x+8$.

Example 3 Find an equation for the tangent line T_N to the cubic curve

$$g(x)=2x^3+6x^2-12x-8$$

at the point $N(1,-12)$ on the curve.

Solution Since Corollary 2 only applies to functions of the form $f(x)=x^3+\alpha x+\beta$, we will first make use of the above remark for this to happen, which.That is, we transform $g(x)=2x^3+6x^2-12x-8$ into the desired form
$f(x)=\frac{1}{a}g\left(x-\frac{b}{3a}\right)=\frac{1}{2}g(x-1)=\frac{1}{2}[2(x-1)^3+6(x-1)^2-12(x-1)+8]$.
That means $f(x)=x^3-9x+4$, which (incidentally!) happens to be the same as the function in our Example 2, so that Corollary 2 could be applied to it. Now, on the other hand that in the process of the above of horizontal shifting an vertical scaling the point $N(1,-12)$ on the graph of the original function $y=g(x)$ will be moved to the point $M(2,-6)$ on the graph of the function $f(x)=x^3-9x+4$. So, at this point we can make use Example 2 and conclude that the equation of the tangent line T_M to graph of the cubic curve defined by $f(x)=x^3-9x+4$ at $M(2,-6)$ is defined by $(x)=3x-12$. Now, as the reader of this note (familiar with function transformations) might agree, we find the required equation of tangent line T_N to the cubic $g(x)=2x^3+6x^2-12x-8$ at $N(1,-12)$ by backwards transforming ations to the linear function $l(x)=3x-12$ to find the equation for T_N. That is, the following back transformed line is what we are looking for,

$$y=2l(x+1)=2[3(x+1)-12]=6x-18.$$

Therefore the required tangent line T_N to the curve $g(x)=2x^3+6x^2-12x-8$ at the point $N(1,-12)$ has equation $y=6x-18$.

I close this short article by mentioning that the above Remark in conjunction with Proposition 2 will enable us to canonically describe all secant segments of any general cubic curve defined by $g(x)=ax^3+bx^2+cx+d$, and conclude a third proposition, but as anyone can guess the algebraic expressions involved in the description won't look handsome , so I think it would be bet to skip this kind messy kind of work !