An Alternative Definition for Derivatives

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The following alternative way to define derivatives that I encountered around 2016 can easily be proved by a simple application of L'Hopial's Rule, provided you assume that a given function $f(x)$ is continuously differentiable at a point $x = a$. But (as seen below) my original proof has the advantage it doesn't assume continuity of $f'(x)$ at $= a$.
A second advantage is that the proof below makes an excellent Exercise/Project for Calculus students at an earlier stage of the course, when they are still dealing with one sided limits, and in particular one sided limits of Newton fractions, as slopes of half tangent lines emanating from the point $(a, f(a))$. An analytical geometric interpretation of this new definition is explained after the proof of the proposition.

Proposition 1 Let $f(x)$ be a differentiable function, and let λ be any positive number. Then the derivative $f'(a)$ of f at any $x = a$ can also be expressed as

$$\lim_{p\to 0} \frac{f(a+\lambda p) - f(a-p)}{(\lambda+1)p} = f'(a) \cdot$$

Proof: It is enough to show that the right hand side limit and the left hand side limit of of the above fraction are both the same as $f'(a)$. I first show that the right hand side limit is $f'(a)$. To this end, write

$$\lim_{p\to 0^+} \frac{f(a+\lambda p) - f(a-p)}{(\lambda+1)p} = \lim_{p\to 0^+} \frac{f(a+\lambda p) - f(a) + f(a) - f(a-p)}{(\lambda+1)p}$$

$$= \lim_{p\to 0^+} \frac{f(a+\lambda p) - f(a)}{(\lambda+1)p} + \lim_{p\to 0^+} \frac{f(a) - f(a-p)}{(\lambda+1)p} =$$

$$= \frac{\lambda}{\lambda+1} \lim_{\lambda p\to 0^+} \frac{f(a+\lambda p) - f(a)}{\lambda p} + \frac{1}{\lambda+1} \lim_{-p\to 0^+} \frac{f(a-p) - f(a)}{-p} =$$

$$= \frac{\lambda}{\lambda+1} f'(a) + \frac{1}{\lambda+1} \lim_{q\to 0^-} \frac{f(x+q) - f(a)}{q} = \frac{\lambda}{\lambda+1} f'(a) + \frac{1}{\lambda+1} f'(a) = f'(a).$$

To show the left hand side limit is also $f'(a)$, for some reason I will use the letter γ (!),

$$\lim_{p\to 0^-} \frac{f(a+\gamma p) - f(a-p)}{(\gamma+1)p} = \lim_{p\to 0^-} \frac{f(a+\gamma p) - f(a) + f(a) - f(a-p)}{(\gamma+1)p}$$

$$= \lim_{p\to 0^-} \frac{f(a+\gamma p) - f(a)}{(\gamma+1)p} + \lim_{p\to 0^-} \frac{f(a) - f(a-p)}{(\gamma+1)p}$$

$$= \frac{\gamma}{\gamma+1} . \lim_{p\to 0^-} \frac{f(a+\gamma p) - f(a)}{\gamma p} + \frac{1}{\gamma+1} . \lim_{-p\to 0^-} \frac{f(a-p) - f(a)}{-p}$$

$$= \frac{\gamma}{\gamma+1} . \lim_{\gamma p\to 0^-} \frac{f(a+\gamma p) - f(a)}{\gamma p} + \frac{1}{\gamma+1} . \lim_{q\to 0^-} \frac{f(a+q) - f(a)}{q}$$

$$= \frac{\gamma}{\gamma+1} f'(a) + \frac{1}{\gamma+1} f'(a) = f'(a)$$

And the proof is complete.

Analytical Geometric Interpretation. In the case of ordinary definition of the derivative of a function $f(x)$ at $x = a$, the tangent line at $(a, f(a))$ is approached by half line secant lines on the curve emanating from the point $(a, f(a))$ itself. But in the above definitioin of the derivative the tangent line at $(a, f(a))$ is approached by secant lines whose ends points are on opposite side of the point $(a, f(a))$ on the curvre. One end has abscissa $x = a - p$ whereas the other end $x = a + \mu p,\ \mu > 0$. For example when $\mu = 1$, and if we choose $p = \frac{1}{n}$,

$$\lim_{n\to\infty} \frac{n}{2}\left[f\left(a+\frac{1}{n}\right) - f\left(a-\frac{1}{n}\right)\right] = f'(a).$$