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Solutions to the "Nice" Coordinate Plane Geometry Problem

In the fall 1999 edition of *Vector*, Gary Tupper presented a specific solution to the following general problem.

Given two points A and B and a line l separating them, find a point P on l such that $\angle APB$ is a right angle.

(See figure 1.)

He then asked whether there still exist two solutions if the required right angle is changed to any other pre-assigned angle, say $\angle APB = 60^\circ$. Also, he asked, "What happens if instead the two points A and B are on the same side of the line l ?"

In the present article, we show that when the line separates the two points, then for any θ ($0^\circ < \theta < 180^\circ$), there are always two points on the line l with $\angle APB = \theta$. However, when the two points are on one side of the line, we present precise conditions under which there may exist four, three, two, one, or no solutions at all.

We also propose another extension: that is the case where one point is on the line and the other one is not. In this case we show that for any θ ($0^\circ < \theta < 90^\circ$), there always exists at least one solution P on l with $\angle APB = \theta$. Also, *Exercise 2* on the last page contains complete refined statements for all values $0^\circ < \theta < 180^\circ$ of the angle θ in this case.

In concluding the statements of the following *Propositions*, we first made use of *Geometer's Sketchpad*. However, once we were convinced of the validity of the results, it was only a matter of time to lay out the algebra.

Extension Problem I

Proposition 1: Let the line $l: y = ax + b$ separate two given points $A(x_1, y_1)$ and $B(x_2, y_2)$. Then for any $0^\circ < \theta < 180^\circ$ there are always two distinct points P_1 and P_2 such that $\angle AP_1B = \angle AP_2B = \theta$.

Case I: $0^\circ < \theta < 90^\circ$:

For a later application in *Case III* we will also show that at the same time, there are two other points Q_1 and Q_2 on l such that

$$\angle AQ_1B = \angle AQ_2B = 180^\circ - \theta$$

Let $M(m, n)$ be the midpoint of the segment AB ; that is $m = \frac{1}{2}(x_1 + x_2)$ and $n = \frac{1}{2}(y_1 + y_2)$.

(See figure 2.)

Let s denote the slope of the segment AB ;

$$s = (y_2 - y_1) / (x_2 - x_1)$$

and let

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

be the distance from A to B .

Let $C_i(u_i, v_i)$, $i = 1, 2$ be the two points with the following coordinates:

$$\begin{aligned} u_1 &= m - \frac{sd \cot \theta}{2\sqrt{1+s^2}} & u_2 &= m + \frac{sd \cot \theta}{2\sqrt{1+s^2}} \\ v_1 &= n + \frac{d \cot \theta}{2\sqrt{1+s^2}} & v_2 &= n - \frac{d \cot \theta}{2\sqrt{1+s^2}} \end{aligned} \quad (1)$$

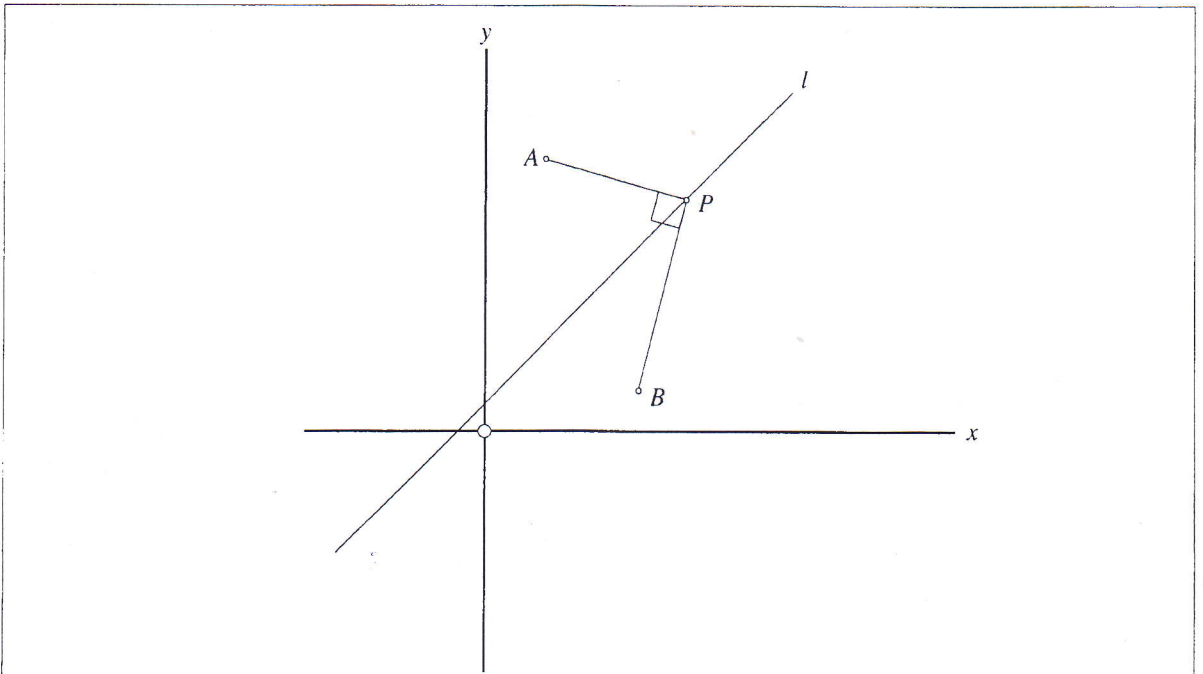


Figure 1

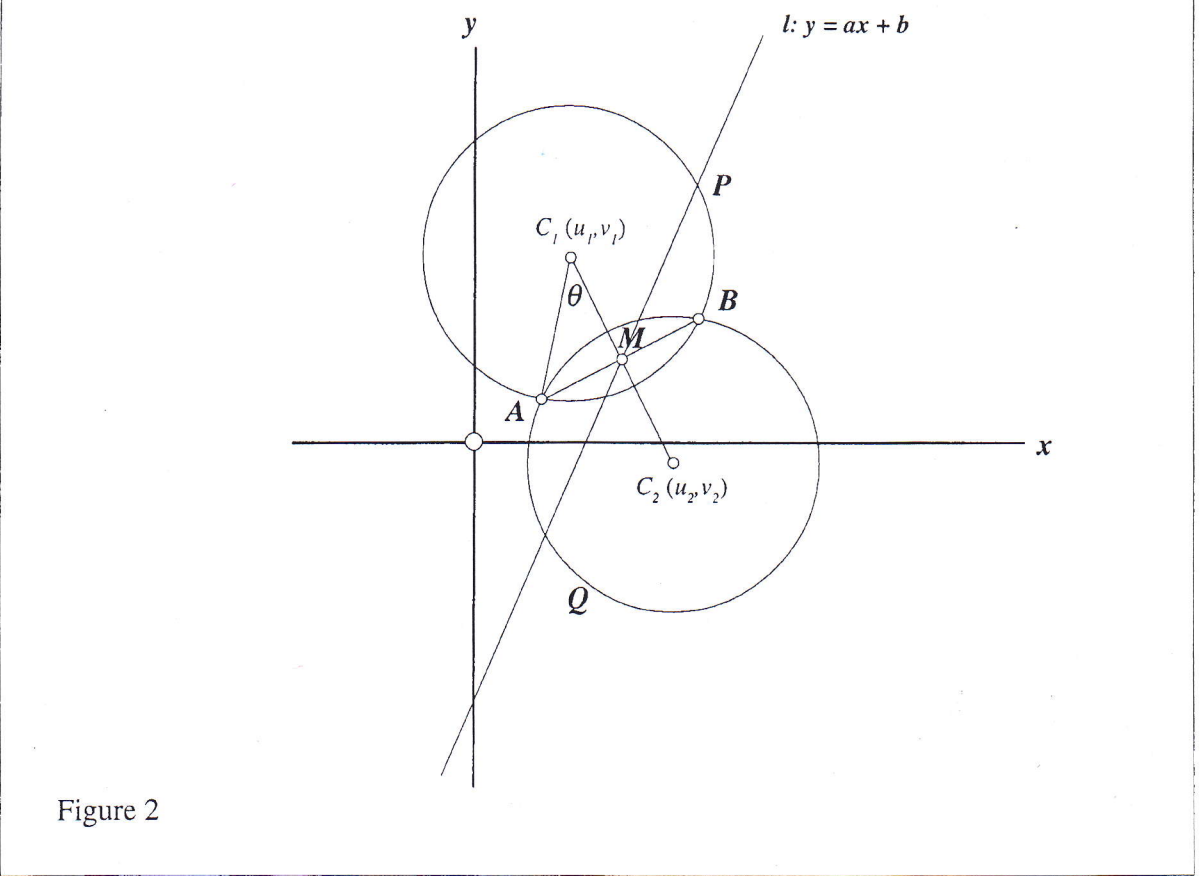


Figure 2

Then, for each selection of $i = 1, 2$; one of the solution points, say $P_i(X_1, Y_1)$, of the system

$$\begin{cases} (X - u_i)^2 + (aX + b - v_i)^2 = d^2 / 4 \sin^2 \theta \\ Y = aX + b \end{cases} \quad (2)$$

will satisfy $\angle AP_i B = \theta$ and the other one $Q_i(X_2, Y_2)$ will, at the same time, satisfy the dual relation $\angle AQ_i B = 180^\circ - \theta$.

To show this claim, we must first show that for each $i = 1, 2$ the first equation in (2) has two distinct solutions for X . To this end we will assume without loss of generality that the line l has equation $y = ax$ with $0 < a$, that l intersects the line segment AB at the origin, and that A and B have coordinates $A(x_1, 0)$ and $B(x_2, 0)$ with $x_1 < 0 < x_2$.

Then $d = x_2 - x_1$, $s = 0$, $m = \frac{1}{2}(x_1 + x_2)$, $n = 0$, and $C_1(u_1, v_1)$ will have coordinates $u_1 = \frac{1}{2}(x_1 + x_2)$ and $v_1 = \frac{1}{2}(x_2 - x_1) \cot \theta$.

Figure 3 (on page 34) shows a sketch of the situation, under the assumed conditions.

Therefore, the first equation in (2) will read as follows,

$$\left[X - \frac{1}{2}(x_1 + x_2) \right]^2 + \left[aX - \frac{1}{2}(x_2 - x_1) \cot \theta \right]^2 = (x_2 - x_1)^2 / 4 \sin^2 \theta$$

This equation is simplified into

$$(1 + a^2)X^2 - [x_1 + x_2 + a(x_2 - x_1) \cot \theta] X + x_1 x_2 = 0 \quad (3)$$

Since $x_1 < 0 < x_2$, the discriminant of this quadratic equation is positive, and therefore the first equation in (2) has two distinct solutions. Let $P(X_1, aX_1)$ and $Q(X_2, aX_2)$ be the two solution points on l . Since in the above equation $x_1 x_2 < 0$, the two solutions of the equation have opposite signs, say $X_2 < 0 < X_1$.

Then $P(X_1, aX_1)$ and $C_1(u_1, v_1)$ are both on the same side of (above) the segment AB . Since by construction, A , B , and P are all on the circle centered at C_1 (and radius $\frac{1}{2}(x_2 - x_1) / \sin \theta$) the inscribed angle $\angle APB$ measures half of the central angle $\angle AQB$.

That is, $\angle APB = \frac{1}{2} \angle AQB = \frac{1}{2}(2\theta) = \theta$. Also since $P(X_1, aX_1)$ and $Q(X_2, aX_2)$ are opposite vertices of a cyclic quadrilateral, we have $\angle AQB = 180^\circ - \theta$.

Similarly, the first equation of the system defined by $C_2(u_2, v_2)$ $C_2(u_2, v_2)$, with coordinates $C_2(\frac{1}{2}(x_1 + x_2), -\frac{1}{2}(x_2 - x_1) / \sin \theta)$ will read as

$$(1 + a^2)X^2 - [x_1 + x_2 - a(x_2 - x_1) \cot \theta] X + x_1 x_2 = 0 \quad (4)$$

This equation will give rises to another pair of P and Q with claimed conditions, and the result follows.

Case II: $\theta = 90^\circ$:

In this case, since $\cot \theta = 0$, the definitions of $C_i(u_i, v_i)$ in (1) show that $C_1(u_1, v_1) = C_2(u_2, v_2) = M(m, n)$, and therefore equations (3) and (4) will be the same. However, in this case, since $180^\circ - \theta = \theta$, we will still have two points P and Q on l satisfying $\angle APB = \angle AQB = 90^\circ$. Therefore, the conclusion of the theorem follows. In this case, since $\sin 90^\circ = 1$, the solutions $P(X_1, Y_1)$ and $Q(X_2, Y_2)$ can be obtained from system (2), upon substituting $\sin 90^\circ = 1$.

Case III: $90^\circ < \theta < 180^\circ$:

In this case, set $\omega = 180^\circ - \theta$. Then $0^\circ < \omega < 90^\circ$, and by *Case I* there exist two distinct points Q_1 and Q_2 on l such that

$$\angle AQ_1B = \angle AQ_2B = 180^\circ - \omega = 180^\circ - (180^\circ - \theta) = \theta$$

Therefore, the proof of the theorem is complete.

Example 1

Given: $l: y = \sqrt{3}x$, points $A(-4, 0)$ and $B(8, 0)$. Then A and B are above and below the line l respectively. Let us choose $\theta = 60^\circ$, say, to be the required angle.

In this case, $M(m, n) = (2, 0)$, $s = 0$, $d = 12$. The two options for $C_i(u_i, v_i)$ are $C_1(2, -2\sqrt{3})$ and $C_2(2, 2\sqrt{3})$.

For selection $C_1(2, -2\sqrt{3})$, equation (3) reads as follows:

$$4X^2 + 8X - 32 = 0, \text{ with solutions } X = -4, 2.$$

Therefore, the two solution points of the system (2) are $P_1(-4, -4\sqrt{3})$ and $Q_1(2, 2\sqrt{3})$. It is now routine to use the *Cosine Law* and show that, indeed, $\angle AP_1B = 60^\circ$ and $\angle AQ_1B = 180^\circ - 60^\circ = 120^\circ$.

The second selection $C(2, 2\sqrt{3})$ will lead to the following specific form of equation (4):

$$4X^2 - 16X - 32 = 0, \text{ with solutions } X = 2 \pm 2\sqrt{3}.$$

Therefore, we obtain another pair of points $P_2(2 + 2\sqrt{3}, 2\sqrt{3} + 6)$ and $Q_2(2 - 2\sqrt{3}, 2\sqrt{3} - 6)$ on l satisfying the same conditions $\angle AP_2B = 60^\circ$ and $\angle AQ_2B = 180^\circ - 60^\circ = 120^\circ$.

Exercise 1 Given: $l: y = (1/\sqrt{3})x$, points $A(-1, \sqrt{3})$, and $B(1, -\sqrt{3})$.

- (a) Show for the selection $\theta = 60^\circ$, that, $M(m, n) = O(0, 0)$, $s = -\sqrt{3}$, $d = 4$, and that the two options for $C_i(u_i, v_i)$ are $C(1, 1/\sqrt{3})$ and $C(-1, -1/\sqrt{3})$.
- (b) Show that for point $C(1, 1/\sqrt{3})$, the system (2) in *Proposition 1* will present the pair of points $P_1(3, \sqrt{3})$ and $Q_1(-1, -1/\sqrt{3})$ on l with conditions $\angle AP_1B = 60^\circ$ and $\angle AQ_1B = 180^\circ - 60^\circ = 120^\circ$.
- (c) Show that for point $C(-1, -1/\sqrt{3})$ the system (2) in *Proposition 1* will present the other pair of points and $P_2(-3, -\sqrt{3})$ and $Q_2(1, 1/\sqrt{3})$ on l with the same conditions $\angle AP_2B = 60^\circ$ and $\angle AQ_2B = 180^\circ - 60^\circ = 120^\circ$.

Extension Problem II

Proposition 2 Let the point $A(x_1, y_1)$ be on the line $l: y = ax + b$, and let $B(x_2, y_2)$ be on one side of l . Then for any $0^\circ < \theta < 90^\circ$, there is at least one point P on l such that $\angle APB = \theta$.

Proof: It is enough to adopt the situation in the proof of *Case I* of *Proposition 1* with $x_1 = 0$. Then equations (3) and (4) in there will read as follows:

$$(1 + a^2)X^2 - [x_2 \pm ax_2 \cot \theta]X = 0. \quad (5)$$

These two equations will give rise to two points P_1 and P_2 on the line l . Since at least one of these points will be on the same side of the x -axis with one of the $C_i(u_i, v_i)$'s, say P_1 , we will have $\angle AP_1B = \theta$.

The other point P_2 will satisfy either $\angle AP_2B = \theta$, or else $\angle AP_2B = 180^\circ - \theta$.

Example 2: Given: $l: y = 2x$, $A = O(0,0)$ and $B(5,0)$. $A = O(0,0)$ and $B(5,0)$. Then A is on l and B is not.

If we now choose $\theta = 45^\circ$ to be the required angle, then the equations (5) will read as $5X^2 - (5 \pm 10)X = 0$, with solutions $X = -1, 3$. These solutions will give rise to the points $P_1(-1, -2)$ and $P_2(3, 6)$ on l . It is now straightforward to use the *Cosine Law* to see that $\angle AP_1B = \angle AP_2B = 45^\circ$.

Because of similarity of the arguments, we will state our last proposition without proof. However, to justify the most interesting case of the proposition, we will present an example with four solution points P on l .

Extension Problem III:

Proposition 3 Let $l: y = ax + b$ be a given line, and assume that two points $A(x_1, y_1)$ and $B(x_2, y_2)$ are both on the same side of the line l . Let $M(m, n)$ be the midpoint of the segment AB ; that is $m = \frac{1}{2}(x_1 + x_2)$ and $n = \frac{1}{2}(y_1 + y_2)$. Let s denote the slope of the segment AB ; that is $s = (y_2 - y_1)/(x_2 - x_1)$. Finally let $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ be the distance from A to B . For any given angle $0 < \theta < 90^\circ$, define $C_i(u_i, v_i), i = 1, 2$, to be the two points with the following coordinates:

$$\begin{aligned} u_1 &= m - \frac{sd \cot \theta}{2\sqrt{1+s^2}} & u_2 &= m + \frac{sd \cot \theta}{2\sqrt{1+s^2}} \\ v_1 &= n + \frac{d \cot \theta}{2\sqrt{1+s^2}} & v_2 &= n - \frac{d \cot \theta}{2\sqrt{1+s^2}} \end{aligned}$$

For each $i = 1, 2$, consider the distance from $C_i(u_i, v_i)$ to the line l ; that is,

$$D_i = \frac{|v_i - au_i - b|}{\sqrt{1+a^2}}$$

Then, assuming without loss of generality that $D_1 \leq D_2$, we distinguish the following cases.

Case (i): $D_1 \leq D_2 < \frac{1}{2} d / \sin \theta$. Then for each $i = 1, 2$ the system

$$\begin{cases} (X - u_i)^2 + (aX + b - v_i)^2 = d^2 / 4 \sin^2 \theta \\ Y = aX + b \end{cases} \quad (6)$$

will have two distinct solutions P_i and Q_i on l , such that $\angle AP_1B = \angle AQ_1B = \theta$.

The other two solutions P_2 and Q_2 will satisfy either the conditions $\angle AP_2B = \angle AQ_2B = \theta$, or else $\angle AP_2B = \angle AQ_2B = 180^\circ - \theta$, depending whether they are on the same, or on the opposite side of the segment AB with $C_2(u_2, v_2)$. Therefore, there may exist as many as four solution points P on l with $\angle APB = \theta$.

Case (ii): $D_1 < D_2 = \frac{1}{2} d / \sin \theta$.

Then for $i = 1$, system (6) will again have two distinct solutions P_1 and Q_1 on l , such that $\angle AP_1B = \angle AQ_1B = \theta$. However, this time for $i = 2$, system (6) will give only one solution Q on l , which will satisfy either $\angle AQB = \theta$ or else $\angle AQB = 180^\circ - \theta$, depending on whether the point Q is on the same, or opposite side of the segment AB with $C_2(u_2, v_2)$. Therefore, all together, there may be two or three solution points P on l with $\angle APB = \theta$.

Case (iii): $D_1 = D_2 = \frac{1}{2} d / \sin \theta$.

Then for each selection $i = 1, 2$, system (6) will have only one solution P_i satisfying $\angle AP_iB = \theta$. Therefore, there will be exactly two solution points P on l with $\angle APB = \theta$.

Case (iv): $D_1 < \frac{1}{2} d / \sin \theta < D_2$.

Then only for $i = 1$, system (6) will have two distinct solutions P_k , $k = 1, 2$ on l satisfying $\angle AP_kB = \theta$. The other system will have no solution on l . Therefore, again, there will be two solution points P on l with $\angle APB = \theta$.

Case (v): $\frac{1}{2} d / \sin \theta = D_1 < D_2$

Then, only for $i = 1$ system (6) will have a single solution P on l , with $\angle APB = \theta$, and the other system will not have any solution. Therefore, this time there will be only one solution P on l satisfying $\angle APB = \theta$.

Case (vi): $\frac{1}{2} d / \sin \theta < D_1 \leq D_2$

Then none of the systems (6) will present any solutions on l , and in this case, there will be no point P on l satisfying $\angle APB = \theta$.

Remark In the above *Extension Problem III* if $\theta = 90^\circ$, then since $\cot \theta = 0$, there will exist only one $C(u, v) = C(m, n)$. In this case, with $\sin \theta = 1$, the statements of the above *Cases (iv)-(vi)* will apply when we drop D_2 from discussion all together.

The following is an example that relates *Case (i)* of *Proposition III*, where there are four solutions points P on the line l with condition that $\angle APB = \theta$ exists.

Example 3: Given $l: y = 2\sqrt{2}x$, and points $A(1,0)$ and $B(5,0)$. Let θ be the angle defined by $\theta = \tan^{-1}(\sqrt{2}/3)$, so that $\cos\theta = 3/\sqrt{11}$ and $\cot\theta = 3/\sqrt{2}$. Here, to the nearest tenth of a degree, we have $\theta = 25.2^\circ$. Then A and B are both below the line l . Also in this case $M(m,n) = (3, 0)$, $s = 0$, $d = 4$, and the two options for $C_i(u_i, v_i)$ are $C(3, 3\sqrt{2})$ and $C(3, -3\sqrt{2})$. For $C(3, -3\sqrt{2})$, system (6) reads as follows:

$$\begin{cases} (X-3)^2 + (2\sqrt{2}X + 3\sqrt{2})^2 = 22 \\ Y = 2\sqrt{2}X \end{cases}$$

The two solutions of this system are $P_1(-\frac{1}{3}, -\frac{2\sqrt{2}}{3})$ and $Q_1(-\frac{5}{3}, -\frac{10\sqrt{2}}{3})$. It is now routine to use the *Cosine Law* and show that $\cos\angle AP_1B = \cos\angle AQ_1B = 3/\sqrt{11}$ and therefore $\angle AP_1B = \angle AQ_1B = \theta$.

The second selection $C(3, 3\sqrt{2})$ will present another system,

$$\begin{cases} (X-3)^2 + (2\sqrt{2}X - 3\sqrt{2})^2 = 22 \\ Y = 2\sqrt{2}X \end{cases}$$

with the solution points

$$P_2(\frac{5+2\sqrt{5}}{3}, \frac{2\sqrt{2}(5+2\sqrt{5})}{3}) \text{ and } Q_2(\frac{5-2\sqrt{5}}{3}, \frac{2\sqrt{2}(5-2\sqrt{5})}{3}).$$

A careful exercise on the *Cosine Law* will show that $\angle AP_2B = \angle AQ_2B = \theta$.

Final Remark: In practice, when an angle θ is pre-assigned, no matter in which of the three positions the points $A(x_1, y_1)$ and $B(x_2, y_2)$ are with respect to the given line $l: y = ax + b$, one can start by finding d , s , and the coordinates of the points $C_i(u_i, v_i)$, $i = 1, 2$, and then try to solve each of the two systems,

$$\begin{cases} (X-u_i)^2 + (aX+b-v_i)^2 = d^2/4 \sin^2\theta \\ Y = aX+b \end{cases}$$

Once a solution point $P(X, Y)$ is obtained for any $i = 1, 2$, then there will be two cases.

- (a) If $[(Y - y_1 - s(X - x_1))][(v_i - y_1 - s(u_i - x_1))] > 0$, then $C_i(u_i, v_i)$ and $P(X, Y)$ will be on the same side of the segment AB and therefore $\angle APB = \theta$.
- (b) If $[(Y - y_1 - s(X - x_1))][(v_i - y_1 - s(u_i - x_1))] < 0$, then $C_i(u_i, v_i)$ and $P(X, Y)$ will be on the opposite sides of the segment AB and therefore $\angle APB = 180^\circ - \theta$.

The following interesting exercise (on *Proposition 2*), can be observed using *Geometer's Sketchpad*.

Exercise 2 Let the point $A(x_1, y_1)$ be on the line $l: y = ax + b$, and let $B(x_2, y_2)$ be on one side of l . Let α be the non-obtuse angle between l and the line segment AB .

Show that:

- (i) For any $0^\circ < \theta < \alpha$ there are always two points P on l such that $\angle APB = \theta$.
- (ii) For any $\alpha \leq \theta < 180^\circ - \alpha$ there is a unique point P on l such that $\angle APB = \theta$.
- (iii) For any $180^\circ - \alpha \leq \theta$ there is no point P on l for which $\angle APB = \theta$.

